Wave Propagation Fundamentals - 1
Propagation of Plane and Spherical Waves
Learning Objectives

Plane waves in fluids and solids

Elastic wave potentials

Spherical waves in fluids and solids
Plane Waves - Fluid

Waves in a fluid satisfy the wave equation

\[ \nabla^2 p - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0 \]

\[ p = \tilde{f}(t - x/c) \] is an arbitrary plane wave"pulse" solution of the wave equation traveling in the + x direction. The function \[ p = \tilde{F}(f) \exp\left[2\pi if \left(\frac{x}{c} - t\right)\right] \] is a harmonic plane wave traveling in the + x direction.

They are simply related through the Fourier Transform:

\[ \tilde{f}(t - x/c) = \int_{-\infty}^{+\infty} \tilde{F}(f) \exp\left[2\pi if \left(\frac{x}{c} - t\right)\right] df \]
we saw previously we can write a plane harmonic wave in different forms:

\[ F(f) \exp[2\pi if(x / c - t)] \]
\[ = F(f) \exp[ik(x - ct)] \]
\[ = F(\omega) \exp[i\omega(x / c - t)] \]

For the last form we can also write

\[ f(t - x / c) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) \exp[i\omega(x / c - t)] d\omega \]
Plane Waves-Fluid

Summary: for a harmonic wave of $\exp(-i\omega t)$ time dependency, a plane wave traveling in the $\pm x$ direction is given by (dropping the common time term)

$$F \exp[\pm 2\pi f x / c]$$
$$= F \exp[\pm i\omega x / c]$$
$$= F \exp[\pm ik x]$$

For a plane wave traveling in an arbitrary direction, $\mathbf{n}$

$$\mathbf{n} \cdot \mathbf{x} = \text{constant}$$

$F \exp[i k \mathbf{n} \cdot \mathbf{x}]$

$= F \exp[i k \cdot \mathbf{x}]$

where $k = k \mathbf{n}$
Waves in a fluid are governed by the scalar wave equation

\[ \nabla^2 p - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0 \]

\[ p = f(t - x \cdot n / c) \]

Waves in an isotropic elastic solid are governed by the vector Navier's equations

\[ \mu \nabla^2 \mathbf{u} + (\lambda + \mu) \nabla(\nabla \cdot \mathbf{u}) - \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = 0 \]

\[ \mathbf{u} \ldots \text{displacement} \]

\[ \lambda, \mu \ldots \text{Lame constants} \]

\[ \rho \ldots \text{density} \]
Navier's equations also have plane wave solutions. There are two types – P-waves and S-waves. These are "bulk" waves.

compressional (P) waves

shear (S) waves

\[ \mu \nabla^2 \mathbf{u} + (\lambda + \mu) \nabla(\nabla \cdot \mathbf{u}) - \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = 0 \]
Plane Waves-Solid

Polarizations

Plane P-waves are longitudinally polarized in the direction of propagation.

Plane S-waves are polarized in the plane perpendicular to the direction of propagation.

Vertically polarized S-waves are called SV-waves while horizontally polarized S-waves are called SH waves.
Plane Waves-Solid

Bulk wave speeds

\[
\begin{align*}
    c_p &= \sqrt{\frac{\lambda + 2\mu}{\rho}} = \sqrt{\frac{E(1-\nu)}{(1+\nu)(1-2\nu)\rho}} \\
    c_s &= \sqrt{\frac{\mu}{\rho}} = \sqrt{\frac{G}{\rho}} = \sqrt{\frac{E}{2(1+\nu)\rho}}
\end{align*}
\]

- \(E\) … Young's modulus
- \(\nu\) … Poisson's ratio
- \(G = \mu\) … Shear Modulus

Compressional wave speed in a bar \(c_{bar} = \sqrt{\frac{E}{\rho}}\)

Compressional wave speed in a plate \(c_{plate} = \sqrt{\frac{E}{(1-\nu^2)\rho}}\)
Although Navier's equations are not wave equations, solutions to Navier's equations can be found in terms of potential functions that do satisfy wave equations

\[ \mu \nabla^2 \mathbf{u} + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) - \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = 0 \]

Plane Waves-Solid

\[ \mathbf{u} = \nabla \phi + \nabla \times \mathbf{\psi} \quad \phi \ldots \text{scalar potential} \]
\[ \mathbf{\psi} \ldots \text{vector potential} \]

\[ \nabla^2 \phi - \frac{1}{c_p^2} \frac{\partial^2 \phi}{\partial t^2} = 0 \quad \text{P-waves} \]

\[ \nabla^2 \mathbf{\psi} - \frac{1}{c_s^2} \frac{\partial^2 \mathbf{\psi}}{\partial t^2} = 0 \quad \text{S-waves} \]
### Table D.1  Acoustical properties of some common materials.

<table>
<thead>
<tr>
<th>Material</th>
<th>Compressional (P-wave) wave speed (m/s $\times 10^3$)</th>
<th>Shear (S-wave) wave speed (m/s $\times 10^3$)</th>
<th>Density $\rho$ (kg/m$^3$ $\times 10^3$)</th>
<th>Impedance (P-wave) (kgm/(m$^2$-s) $\times 10^6$)</th>
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<tbody>
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<td>Air</td>
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<td>27.3</td>
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<td>101.0</td>
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<tr>
<td>Water</td>
<td>1.48</td>
<td>--</td>
<td>1.00</td>
<td>1.48</td>
</tr>
</tbody>
</table>
Plane Waves-Solid

Elastic Wave Potentials

\[ \mathbf{u} = \nabla \phi + \nabla \times \psi \]

For two-dimensional waves

\[ \phi = \phi(x, y, t) \]
\[ \psi_z = \psi(x, y, t), \psi_x = \psi_y = 0 \]

Displacements

\[ u_x = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \]
\[ u_y = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} \]
\[ u_z = 0 \]
\[ \kappa = \frac{c_p}{c_s} \]

Stresses

\[ \tau_{xx} = \mu \left[ \kappa^2 \nabla^2 \phi + 2 \left( \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y^2} \right) \right] \]
\[ \tau_{yy} = \mu \left[ \kappa^2 \nabla^2 \phi - 2 \left( \frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial x^2} \right) \right] \]
\[ \tau_{xy} = \mu \left[ 2 \frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2} \right] \]
\[ \tau_{zz} = \nu \left[ \tau_{xx} + \tau_{yy} \right], \ \tau_{xz} = \tau_{yz} = 0 \]
Plane Waves-Solid

Plane wave solutions

P-waves \( k_p = \omega / c_p \)

\[ \phi = \Phi \exp(ik_p x - i \omega t) \] potential

\[ u_x = U_x \exp(ik_p x - i \omega t) \] displacement

\[ v_x = V_x \exp(ik_p x - i \omega t) \] velocity

\[ \tau_{xx} = T_{xx} \exp(ik_p x - i \omega t) \] stress

\[ U_x = ik_p \Phi, \quad V_x = -i \omega U_x, \]

\[ T_{xx} = -\rho c_p V_x \]
Plane Waves-Solid

S-waves

\[ k_s = \omega / c_s \]
\[ s = e_x \times t \]

\[ \psi = \Psi t \exp(ik_s x - i\omega t) \] potential

\[ u = U_s s \exp(ik_s x - i\omega t) \] displacement

\[ v = V_s s \exp(ik_s x - i\omega t) \] velocity

\[ \tau_{xs} = T_{xs} \exp(ik_s x - i\omega t) \] stress

\[ U_s = ik_s \Psi, \quad V_s = -i\omega U_s, \]

\[ T_{xs} = -\rho c_s V_s \]
Spherical waves arise physically from "point" sources. If a point source emits waves uniformly in all directions, then we expect the waves to depend only on the radial distance, $r$, from the point source.
Spherical Waves-Fluid

Consider spherical harmonic waves of $\exp(-i\omega t)$ time dependency. The equation of motion of the fluid and the wave equation are given by

$$-\nabla p = -i\omega \rho \mathbf{v}$$

$$\nabla^2 p + \frac{\omega^2}{c^2} p = 0$$

In terms of the distance, $r$, from the source, in spherical coordinates these equations are

$$\frac{\partial p}{\partial r} = i\omega \rho \mathbf{v}_r,$$

$$\frac{\partial^2 p}{\partial r^2} + 2 \frac{\partial p}{r \partial r} + \frac{\omega^2}{c^2} p = 0$$
Spherical Waves-Fluid

There are two solutions of the wave equation of the form

\[ p = \frac{A}{r} \exp(ikr) + \frac{B}{r} \exp(-ikr) \]

\[ k = \frac{\omega}{c} \]

The first term represents a wave traveling in the outward radial direction while the second term represents a wave converging on the source. Since the source only generates outward going waves, we must set \( B = 0 \). The pressure and radial velocity in the outgoing wave are then

\[ p = \frac{A}{r} \exp(ikr) \]

\[ v_r = \frac{A}{\rho c} \left[ 1 - \frac{1}{ikr} \right] \frac{\exp(ikr)}{r} \]
Spherical Waves-Fluid

In many cases we are only interested in the waves many wavelengths from the source, in which case $kr \gg 1$ and we can write approximately

\[ p = A \frac{\exp(i kr)}{r} \]
\[ \nu_r = \frac{A}{\rho c} \frac{\exp(i kr)}{r} \]

Compare this to a plane wave traveling in the $+z$ direction where

\[ p = A \exp(i k z) \]
\[ \nu_z = \frac{A}{\rho c} \exp(i k z) \]
Spherical Waves-Elastic Solid

Spherical waves from point sources in an elastic solid have a more complex structure in general, but for \( kr \gg 1 \) they are similar to that of a fluid:

\[
\mathbf{u} = A \frac{\exp(ik_p r)}{r} + B \frac{\exp(ik_s r)}{r}
\]

A \( \square \) \( \mathbf{e}_r \)

B \( \perp \) \( \mathbf{e}_r \)

**S-wave**

**P-wave**