Transducer Sound Radiation
**Learning Objectives**

Planar Immersion Transducer
- on-axis near field - far field
- radiation into solid-normal incidence, plane interface
- diffraction correction

Spherically focused transducer
- on axis field
- focal spot size
- radiation into solid-normal incidence, plane interface
- diffraction correction
Learning Objectives (continued)

Contact P-wave transducer on a solid
  wave types present
  directivity functions

Angle beam shear wave transducer

Overview of beam theories
Ultrasonic Beam Models

Plane piston transducer radiating into a fluid

\[ dS \]

\[ \omega(\omega) = \frac{-i \omega \rho v_z}{2\pi} \exp(ikr) \]

Can show that each area element that is in motion acts like a source of spherical waves:
Adding up all such sources over the face of the transducer gives the Rayleigh-Sommerfeld Integral

\[ p(x, y, z, \omega) = \frac{-i\omega\rho}{2\pi} \int_{S} \frac{v_z(x', y', \omega) \exp(ikr)}{r} \, dS(y) \]

If we let \( v_z(x', y', \omega) = v_0(\omega) \) (piston model)

\[ \Rightarrow \quad p(x, y, z, \omega) = \frac{-i\omega\rho v_0(\omega)}{2\pi} \int_{S} \frac{\exp(ikr)}{r} \, dS(y) \]
\[ p(\mathbf{x}, \omega) = \frac{-i \omega \rho v_0}{2\pi} \int_{S} \frac{\exp(ikr)}{r} \, dS(y) \]

For on-axis response of a circular transducer of radius, a

It can be shown that the area element can be written as \( dS = r \, dr \, d\phi \)

\[ p(z, \omega) = \rho cv_0 \left[ \exp(ikz) - \exp(ik\sqrt{a^2 + z^2}) \right] \]

direct wave

edge waves
Direct and edge waves as seen for a pulsed transducer
Near field distance $N = \frac{a^2}{\lambda}$

Maxima: $z = \frac{N}{(2m+1)}$  $m = 0,1,2,\ldots$
Minima: $z = \frac{N}{2n}$  $n = 1,2,3,\ldots$

Example: for a 5 MHz, 1/2 in. diameter transducer radiating into water
$N = 5$ in. (approx.)
Paraxial approximation: \( a/z << 1 \)

\[
\sqrt{a^2 + z^2} \approx z \left( 1 + \frac{a^2}{2z^2} + \ldots \right)
\]

On-axis pressure:

\[
p(z, \omega) = \rho c v_0 \exp(ikz) \left| 1 - \exp \left( \frac{ika^2}{2z} \right) \right|
\]

plane wave \( C_1(a, \omega, z) \)

diffraction correction
on-axis pressure:

\[ \frac{|p|}{\rho c v_0} \]

exact result

\[ \frac{|p|}{\rho c v_0} \]

paraxial result
function p = on_axis(zN, A,c,F)
% exact on axis pressure from a piston source
% radiating into a fluid. A is radius in mm, c the
% wavespeed of the fluid in m/sec, F the frequency in MHz,
% zN is the distance in the fluid divided by the near field
% distance a^2/lamba (lamba is the wavelength)
al= 1000*A*F/c;    % a/lamba
ka = 2*pi*al;    % ka for the transducer
kz = ka*al*zN;
ke = 2*pi*(al^2).*sqrt(zN.^2 + (1/al)^2);
p = exp(i*kz) - exp(i*ke);
function p = par_on_axis(zN, A,c,F)
% paraxial axis pressure from a piston source
% radiating into a fluid. A is radius in mm, c the
% wavespeed of the fluid in m/sec, F the frequency in MHz,
% zN is the distance in the fluid divided by the near field
% distance a^2/lambda
al= 1000*A*F/c;  % a/lambda
ka = 2*pi*al;     % ka for the transducer
kz = ka*al*zN;
ke = ka./(2*al.*zN);
p = exp(i*kz).*(1 - exp(i*ke));
MAT> z = linspace(.2, 4,500);
MAT> p = on_axis(z,6.35,1500,5);
MAT> plot(z, abs(p))
MAT> xlabel('z/N')

MAT> p = par_on_axis(z, 6.35, 1500, 5);
MAT> plot(z, abs(p))
MAT> xlabel('z/N')
On-axis response at normal incidence to a plane interface (paraxial approximation)

\[
\mathbf{u}(x, \omega) = \frac{v_0}{-i\omega} T_{12}^{p;p} d_p \exp \left( i k_{p1} z_1 + k_{p2} z_2 \right) \left[ 1 - \exp \left( \frac{ik_{p1}a^2}{2\tilde{z}} \right) \right]
\]

transmission coefficient
(at normal incidence)
(velocity/velocity)

\[
\tilde{z} = z_1 + \frac{c_{p2}}{c_{p1}} z_2
\]

same diffraction correction expression as for a single fluid
"virtual" point where the edge wave would arrive on-axis in the solid if $c_{p2} = c_{p1}$

\[
\varepsilon = d_2 \sin \theta_{p2} = d \sin \theta_{p1} \quad \text{so} \quad d = \frac{\sin \theta_{p2}}{\sin \theta_{p1}} d_2 = \frac{c_{p2}}{c_{p1}} d_2
\]

which gives, in the paraxial approximation

\[
\frac{a^2}{\tilde{z}} \approx \frac{a^2}{d_1 + d} \approx \frac{a^2}{d_1 + \frac{c_2}{c_1} d_2} \approx \frac{a^2}{z_1 + \frac{c_2}{c_1} z_1}
\]
Far-field beam of a planar piston transducer

The far-field is usually defined as $z > 3N$ - also called the "spherical wave region"

$y = (x, y, 0)$

$$r = \sqrt{(x - y) \cdot (x - y)}$$
$$= \sqrt{(Re - y) \cdot (Re - y)}$$
$$\approx R\sqrt{1 - 2e \cdot y / R}$$
$$\approx R - e \cdot y$$

$$p(x, \omega) = \frac{-i\omega \rho v_0}{2\pi} \frac{\exp(ikR)}{R} \int_S \exp(-ike \cdot y) dxdy$$
Define the 2-D spatial Fourier transform of $\Theta$, where

$$\Theta = \begin{cases} 
1 & \text{in } S \\
0 & \text{otherwise}
\end{cases}$$

as

$$F(e_x, e_y, \omega) = \frac{1}{(2\pi)^2} \iint_S \exp(-ip_x x - ip_y y) \, dx \, dy$$

Then the far field pressure can be written as

$$p(x, \omega) = -2\pi i \omega \rho v_0 F(e_x, e_y, \omega) \frac{\exp(ikR)}{R}$$

angular beam profile  spherical wave
Rectangular Piston Transducer

\[ p(x, \omega) = -2\pi i\omega \rho v_0 F \frac{\exp(ikR)}{R} \]

\[ F(e_x, e_y, \omega) = \frac{ab}{(2\pi)^2} \frac{\sin\left(\frac{kbe_x}{2}\right) \sin\left(\frac{kae_y}{2}\right)}{\left(\frac{kbe_x}{2}\right) \left(\frac{kae_y}{2}\right)} \]

In spherical coordinates

\[ e_x = \sin \theta \cos \phi \]
\[ e_y = \sin \theta \sin \phi \]
Example far-field pattern of a rectangular transducer
Circular Piston Transducer

\[ e_\rho = \sqrt{e_x^2 + e_y^2} = \sin \theta \]

\[ p(x, \omega) = -2\pi i \omega \rho \nu_0 F \frac{\exp(ikR)}{R} \]

\[ F(e_x, e_y, \omega) = \frac{a^2}{2\pi} \frac{J_1(ke_\rho a)}{(ke_\rho a)} \]

\[ \frac{|p|}{\rho c \nu_0} \]

\[ z = 3N \]

\[ z = 6N \]
function [p, rho] = far_field(ang, A, c, F, RN)
% far_field computes the normalized far field pressure
% for a circular piston (omitting the exp(ikR) phase term)
% A is the radius of the transducer in mm, c the wavespeed
% in m/sec, F the frequency in MHz, and RN is
% the normalized radial distance in near field units.
% rho is the transverse distance (normal to z) in mm
ka = 2*pi*(1000*A*F/c);
al= 1000*A*F/c;
x = ka*sin(ang*pi/180);
rho =RN*(A*al)*sin(ang*pi/180);
p = -i*(ka/(al*RN))*besselj(1,x)./(x+eps*(x ==0));
Spherically Focused Piston Transducer Radiating Into a Fluid

O’Neil Model

uniform velocity, $v_0$

$S_f$ ... spherical surface

$$p(x, \omega) = \frac{-i \omega \rho v_0}{2\pi} \int_{S_f} \frac{\exp(ikr)}{r} \, dS(y)$$
For \( \mathbf{x} \) on the central axis

\[
dS = r \, dr \, d\phi / q_0 \quad q_0 = 1 - z / R_0
\]

\[
p(\mathbf{x}, \omega) = \frac{\rho_{cv_0}}{q_0} \left[ \exp(ikz) - \exp(ikr_e) \right]
\]

\[
r_e = \sqrt{(z - h)^2 + a^2} \quad h = R_0 - \sqrt{R_0^2 - a^2}
\]
a = 6.35 mm
R₀ = 76.2 mm
f = 10 MHz
c = 1480 m

|p| / ρcv₀

on-axis pressure versus z/R₀:

True focus
function p = focused_on_axis(zR, A, c, F, R)
% on axis pressure of a spherically focused probe
% as a function of the normalized distance, zR = z/R
%A, radius of the transducer in mm. R, focal length in mm.
%c, the wave speed in m/sec, and F the frequency in MHz
al = 1000 * A * F / c;
ka = 2 * pi * al;
zN = (R / A) * (1 / al) * (zR);
kz = ka * al * zN;
kR = 2000 * pi * F * R / c;
kh = kR - sqrt(kR^2 - ka^2);
kre = sqrt((kz - kh).^2 + ka^2);
p = (exp(i * kz) - exp(i * kre))./(1 - kz ./ kR);

MAT> z = linspace(.2, 4, 500);
MAT> p = focused_on_axis(z, 6.35, 1480, 10, 76.2);
MAT> plot(zr, abs(p))
MAT> xlabel('z/R')
Paraxial Approximation

\[ r_e \approx z + \frac{a^2 q_0}{2z} \]
\[ q_0 = 1 - \frac{z}{R_0} \]

on-axis pressure:

\[ p(z, \omega) = \rho c v_0 \exp(ikz) \left\{ \frac{1}{q_0} \left[ 1 - \exp\left( \frac{ika^2 q_0}{2z} \right) \right] \right\} \]

plane wave \hspace{1cm} \text{diffraction correction} \hspace{1cm} C_1(a, z, R_0, \omega)
on-axis pressure

exact \[
\frac{|p|}{\rho cv}
\]

paraxial \[
\frac{|p|}{\rho cv_c}
\]
function p = par_focused_on_axis(zR, A,c,F,R)
% on axis pressure of a spherically focused probe, paraxial approx.
% as a function of the normalized distance, zR = z/R
%A, radius of the transducer in mm. R , focal length in mm.
%c, the wave speed in m/sec, and F the frequency in MHz
al=1000*A*F/c;
ka=2*pi*al;
zN=(R/A)*(1/al)*(zR);
kz=ka*al*zN;
kR=2000*pi*F*R/c;
qo=1-kz./kR;
p = (1-exp(i*ka*(A/R)*qo./(2*zR)))./qo;
MAT> z=linspace(.2,4,500);
MAT> p = par_focused_on_axis(z,6.35,1480,10,76.2);
MAT> plot(z, abs(p))
MAT> xlabel('z/R')
Another way to model focusing (in the paraxial approximation)

suppose on a planar aperture we have a spherical wave propagating (generated by a lens, for example)

then on the aperture we have a phase given approximately in the paraxial approximation \((\rho_0 / R_0 \ll 1)\) by

\[
\exp(-ik[r_s - R_0]) = \exp\left[-ik\left[\sqrt{\rho_0^2 + R_0^2} - R_0\right]\right] \\
\approx \exp\left(-ik\frac{\rho_0^2}{2R_0}\right)
\]
Thus, suppose we use a Rayleigh-Sommerfeld model for a planar transducer and place this phase (in the paraxial approximation) in the integral:

\[ p(x, \omega) = \frac{-i\omega \rho v_0(\omega)}{2\pi} \iint_s \exp(-ik\rho_0^2/2R_0) \frac{\exp(ikr)}{r} dS \]

Using the paraxial approximation and evaluating this integral exactly for \( x \) on the transducer axis gives for a circular transducer of radius \( a \):

\[ p(z, \omega) = \frac{\rho cv_0 \exp(ikz)}{q_0} \left[ 1 - \exp(ika^2 q_0 / 2z) \right] \]

Similarly, off-axis values will also represent those from a focused transducer.
Wave field in the plane at the geometric focus of a spherically focused transducer

\[ p(x, \omega) = -i \omega \rho v_0 a^2 \frac{\exp(ik\bar{R}_0)}{\bar{R}_0} J_1 \left( \frac{kay}{\bar{R}_0} \right) \]

\[ W_f \big|_{6\,dB} = 4.43 \frac{R_0}{ka} = 1.41 \lambda F \]

\( \lambda \) ... wavelength

\( F = R_0 / 2a \) ... transducer F number
On-axis response at normal incidence to an interface (paraxial approximation)

\[ \mathbf{u}(x, \omega) = \frac{v_0}{-i\omega} T_{12}^{P,P} d_p \exp\left(ik_p z_1 + k_p z_2\right) \left\{ \frac{1}{q_0} \left[ 1 - \exp\left( \frac{ik_p a^2 \tilde{q}_0}{2\tilde{z}} \right) \right] \right\} \]

\[ \tilde{q}_0 = 1 - \frac{\tilde{z}}{R_0} \]

\[ \tilde{z} = z_1 + \frac{c_{p2}}{c_{p1}} z_2 \]

Displacement

Transmission coefficient (velocity/velocity)

Same diffraction correction expression as for a fluid
Contact P-wave Transducer Model

For bulk waves

\[
\mathbf{u}(\mathbf{x}', \omega) = \frac{p_0}{2\pi \rho_1 c_{s1}^2} \int K_s(\theta') d_1^s \frac{\exp(ik_{s1}D)}{D} dS(\mathbf{x}'') \\
+ \frac{p_0}{2\pi \rho_1 c_{p1}^2} \int K_p(\theta') d_1^p \frac{\exp(ik_{p1}D)}{D} dS(\mathbf{x}'')
\]
Directivity functions

\begin{align*}
K_p (\theta') &= \frac{\cos \theta' \kappa_1^2 (\kappa_1^2 / 2 - \sin^2 \theta')}{2 G(\sin \theta')}, \\
K_s (\theta') &= \frac{\kappa_1^3 \cos \theta' \sin \theta' \sqrt{1 - \kappa_1^2 \sin^2 \theta'}}{2 G(\sin \theta')}, \\
G(x) &= \left(x^2 - \kappa_1^2 / 2\right)^2 + x^2 \sqrt{1 - x^2} \sqrt{\kappa_1^2 - x^2}.
\end{align*}

\[ \kappa_1 = \frac{c_{p1}}{c_{s1}} \]
function [kp,ks] = directivity(ang, cp, cs)
% computes the directivity functions for a p-wave contact %transducer. ang is angle in degrees, cp, cs are p- and s-wave %speeds
k = cp/cs;
angr = ang*pi/180;
x = sin(angr);
c = cos(angr);
g = (x.^2 - k^2/2).^2 + x.^2.*sqrt(1 - x.^2).*sqrt(k^2 - x.^2);
kp = c.*(k^2).*(k^2/2 - x.^2)./(2.*g);
ks = (k*x <1).*c.*(k^3).*x.*sqrt(1 - k^2.*x.^2)./(2.*g);
MAT> x = linspace(0,90,200);
MAT> [kp,ks] = directivity(x, 5900, 3200);
MAT> plot(x, kp)
MAT> hold on
MAT> plot(x, ks)
MAT> xlabel('angle, degrees')
For θ' small \( K_p = 1, K_s = 0 \)

\[
\mathbf{u}(\mathbf{x}', \omega) = \frac{p_0 n}{2 \pi \rho_1 c_{p1}^2} \int \frac{\exp(ik_{p1}D)}{D} dS
\]

Full set of waves:

- \( D^p \) … Direct P-wave
- \( E^p \) … Edge P-wave
- \( E^s \) … Edge S-wave
- \( H \) … Head wave
- \( R \) … Rayleigh wave

integral contains direct and edge P-waves
Angle Beam Shear Wave Transducer Model

Can replace elastic wedge by equivalent "fluid" (neglect shear waves)

(small for incident P-wave beyond critical angle)
Ultrasonic Beam Models

Numerically Intense Models
  EFIT - Langenberg
  Finite Elements - Lord
  Boundary Elements - Rizzo
  Edge Elements - Schmerr, Lerch

Surface Integral Models
  Generalized Point Source - Spies
  Rayleigh-Sommerfeld + High Freq. Asymptotics
  - Schmerr, Lhemery, others

Line Integral Models
  Boundary Diffraction Wave - Schmerr, Lerch
Ultrasonic Beam Models

Other Basis Function Models
Gauss-Hermite Models - Thompson, Gray, Newberry, Minachi, Margetan

Multi- Gaussian Models
Minachi, Spies, Schmerr and Rudolph, Cerveny (Seismology)
Ultrasonic Beam Models

A few references – mostly paraxial models


Ultrasonic Beam Models


