Ultrasonic System Measurement Model - II
Learning Objectives

Definition of a Measurement Model

Three Types of Measurement Models
  Auld Model
  Thompson-Gray
  Model for 2D scatterers

Verifying the Modeling Assumptions
and Measurements Needed
What is an Ultrasonic Measurement Model? A model that expresses the output voltage, $v_R(t)$, received from a flaw in an ultrasonic testing configuration in a form that explicitly states the modeling or measurement parameters needed to obtain that voltage (immersion shown only as a specific example).

All the measurement models discussed use $V_R(\omega)$, the frequency components of $v_R(t)$.

$$V_R(\omega) \overset{FFT}{\leftrightarrow} v_R(t)$$
Electroacoustic Measurement Model - 2002


\[ V_R(\omega) = t_G(\omega)t_R(\omega)V_i(\omega)t_A(\omega) \]
\[ = s(\omega)t_A(\omega) \]

\( V_i(\omega) \) \hspace{1cm} \text{pulser} \hspace{1cm} \text{cabling} \hspace{1cm} \text{sending} \hspace{1cm} \text{transducer} \hspace{1cm} \begin{array}{c} t_A(\omega) = ? \\ \text{propagation, flaw scattering} \end{array} \hspace{1cm} \text{receiving} \hspace{1cm} \text{transducer} \hspace{1cm} \text{cabling} \hspace{1cm} \text{receiver} \hspace{1cm} V_R(\omega) \]
Auld Electromechanical Reciprocity Relation

Auld (1979), using reciprocity relations and the solutions to two specific problems, developed an explicit relation that has been widely used.


The two problems:

- **Problem (1)**
  - Transducer A firing,
  - Flaw present

- **Problem (2)**
  - Transducer B firing,
  - Flaw absent
Ultrasonic Measurement Models

Auld Electromechanical Reciprocity Relation

\[ \delta \Gamma = \frac{1}{4P} \int_{S_f} \left( \mathbf{t}^{(1)} \cdot \mathbf{v}^{(2)} - \mathbf{t}^{(2)} \cdot \mathbf{v}^{(2)} \right) \, dS \]

\( \delta \Gamma \)… change in cable transmission coefficient due to the presence of the flaw

\( P \)… power delivered by the transducer

\( \mathbf{t}^{(1)}, \mathbf{v}^{(1)} \)… stress vector and velocity for problem (1): A firing and flaw present

\( \mathbf{t}^{(2)}, \mathbf{v}^{(2)} \)… stress vector and velocity for problem (2): B firing and flaw absent
Auld Electromechanical Reciprocity Relation

Auld's assumptions:
1) linear, reciprocal electromechanical system (harmonic disturbances)
2) fundamental TEM mode propagating in the cable

Since we expect \( V_R(\omega) \propto \delta \Gamma(\omega) \) Auld's relation "essentially" is an ultrasonic measurement model
An Alternative Reciprocity Relation

In 1998, Dang, Schmerr, and Sedov obtained a completely explicit model relationship using a purely mechanical reciprocity relationship plus several other relatively weak assumptions.

\[ V_R(\omega) = \frac{s(\omega)}{Z_r^A(\omega)v^{(1)}_A(\omega)v^{(2)}_B(\omega)} \int_{S_f} \left( t^{(1)} \cdot v^{(2)} - t^{(2)} \cdot v^{(1)} \right) dS \]
Ultrasonic Measurement Models

\[
V_R(\omega) = \frac{s(\omega)}{Z_r^A(\omega)} \cdot \frac{v_A^{(1)}(\omega)}{v_B^{(2)}(\omega)} \int_{S_f} \left( t^{(1)} \cdot v^{(2)} - t^{(2)} \cdot v^{(1)} \right) dS
\]

- \( s(\omega) \) … system function (pulser, cabling, receiver)
- \( v_A^{(1)}(\omega) \) … average velocity over the face of transducer A in problem (1)
- \( v_B^{(2)}(\omega) \) … average velocity over the face of transducer B in problem (2)
- \( Z_r^A(\omega) \) … radiation impedance of transducer A in problem (1)
Ultrasonic Measurement Models

\[ V_R(\omega) = \frac{S(\omega)}{Z_r^A(\omega) \nu_A(\omega) \nu_B(\omega)} \int_{S_f} \left( \mathbf{t}^{(1)} \cdot \mathbf{v}^{(2)} - \mathbf{t}^{(2)} \cdot \mathbf{v}^{(1)} \right) dS \]

assumptions used to obtain this explicit model

1) linear, reciprocal acoustic and elastic media (harmonic disturbances)
2) we also assumed the output velocity for the transducers can be written in a separable form. For transducer A, for example

\[ v_A^{(1)}(x, \omega) = v_A^{(1)}(\omega) f(x) \]

Here, the "forces" are defined in terms of weighted pressure integrals over the transducer face, using the same functions used to describe the velocity. For example, for transducer A

\[ F_t(\omega) = \int_{S_A} p(x, \omega) f(x) dS(x) \]

For a piston transducer \( f = 1 \).
These two assumptions lead to an explicit expression for the acoustic/elastic transfer function:

\[ t_A(\omega) = \frac{F_B(\omega)}{F_t(\omega)} = \frac{1}{Z_r^A(\omega) v_A^{(1)}(\omega) v_B^{(2)}(\omega)} \int_{S_f} \left( \mathbf{t}^{(1)} \cdot \mathbf{v}^{(2)} - \mathbf{t}^{(2)} \cdot \mathbf{v}^{(1)} \right) dS \]
3) we also assume that the pulser, receiver, transducers and cabling can be replaced by equivalent linear, time-shift invariant (LTI) systems, i.e.

\[
F_t(\omega) = t_G(\omega)V_{in}(\omega)
\]

\[
V_R(\omega) = t_R(\omega)F_B(\omega)
\]

Then, we can write the reciprocity relationship in the form

\[
V_R(\omega) = \frac{t_R(\omega)t_G(\omega)V_{in}(\omega)}{Z_r^A(\omega)v_A^{(1)}(\omega)v_B^{(2)}(\omega)} \int_{S_f} \left( t^{(1)} \cdot v^{(2)} - t^{(2)} \cdot v^{(1)} \right) dS
\]

or, finally

\[
V_R(\omega) = \frac{s(\omega)}{Z_r^A(\omega)v_A^{(1)}(\omega)v_B^{(2)}(\omega)} \int_{S_f} \left( t^{(1)} \cdot v^{(2)} - t^{(2)} \cdot v^{(1)} \right) dS
\]
A General Ultrasonic Measurement Model

\[
V_R(\omega) = \frac{s(\omega)}{Z_r(\omega) v_A^{(1)}(\omega) v_B^{(2)}(\omega)} \int_{S_f} \left( t^{(1)} \cdot v^{(2)} - t^{(2)} \cdot v^{(1)} \right) dS
\]

assumptions

1) linear, reciprocal acoustic and elastic media (harmonic disturbances)

2) the output velocity for the transducers can be written in a separable form

3) the pulser, receiver, transducers and cabling can be replaced by equivalent linear, time-shift invariant (LTI) systems.
4) if we also assume that the transducers behave as piston radiators at high frequency then

\[ Z_r^A(\omega) = \rho_1 c_1 S_A \]

and our explicit model becomes, finally

\[
V_R(\omega) = \frac{S(\omega)}{\rho_1 c_1 S_A v_A^{(1)}(\omega) v_B^{(2)}(\omega)} \int_{S_f} \left( t^{(1)} \cdot v^{(2)} - t^{(2)} \cdot v^{(1)} \right) dS
\]
5) now, assume the incident waves are quasi-plane waves in the vicinity of the flaw, e.g.

\[
v_{j}^{(1)} \bigg|_{inc} = V^{(1)} (x_1, x_2, x_3, \omega) d_j^{(1)} \exp(i k_2 e^{(1)} \cdot x) \\
v_j^{(2)} = V^{(2)} (x_1, x_2, x_3, \omega) d_j^{(2)} \exp(i k_2 e^{(2)} \cdot x)
\]

and also define normalized amplitudes of these quasi-plane waves as

\[
\hat{V}^{(1)} = \frac{V^{(1)}}{v_A^{(1)}(\omega)} \\
\hat{V}^{(2)} = \frac{V^{(2)}}{v_B^{(2)}(\omega)}
\]
Then the form of the measurement model becomes

\[ V_R(\omega) = s(\omega) \left[ \frac{4\pi \rho_2 c_2}{-i k_2 \rho_1 c_1 S_A} \right] \int_{S_f} \hat{V}^{(1)} \hat{V}^{(2)} A(\hat{v}_j^{(1)}, \hat{\tau}_{ij}^{(1)}) dS \]

velocity fields in problems (1),(2) due to unit average velocities on the transducer faces

scattered wave fields from the flaw
The flaw scattering term is rather complex:

\[
A\left(\tilde{\nu}_{ij}^{(1)}, \tilde{\tau}_{ij}^{(1)}\right) = \frac{1}{4\pi \rho_2 c_2^2} \left(\tilde{\tau}_{ij}^{(1)} \mu_{ij}^{(2)} - C_{ijkl} d_{kl}^{(2)} \left(\mathbf{e}_{l}^{(2)} / c_2\right) \tilde{\nu}_{i}^{(1)}\right) n_j \exp\left(ik_2 \mathbf{e}^{(2)} \cdot \mathbf{x}\right)
\]

\(\tau_{ij}^{(1)}, \nu_{ij}^{(1)}\) stresses and velocity components in problem (1)

\(e_{j}^{(2)}, d_{j}^{(2)}\) components of the incident wave direction and polarization in problem (2)

\(n_j\) components of the unit outward normal to the flaw surface

\(C_{ijkl}\) elastic constants tensor

the normalized fields \(\tilde{\nu}_{ij}^{(1)}, \tilde{\tau}_{ij}^{(1)}\) are defined as velocity and stress in problem (1) due to an incident wave of unit displacement amplitude

\[
\tilde{\nu}_{ij}^{(1)} = \frac{-i\omega \nu_{ij}^{(1)}}{V^{(1)}}
\]

\[
\tilde{\tau}_{ij}^{(1)} = \frac{-i\omega \tau_{ij}^{(1)}}{V^{(1)}}
\]

\(U^{(1)} = 1\)

\(\tilde{\tau}_{ij}, \tilde{\nu}_{ij}\)
3-D scattering amplitude

\[ u^{\text{scatt}}(y, \omega) = U_0 \frac{A(e_i^\beta; e_s^P)}{r_s} \exp(ik_p r_s) + U_0 \frac{A(e_i^\beta; e_s^S)}{r_s} \exp(ik_s r_s) \]

scattered displacement from a flaw

P-wave

S-wave

incident wave with polarization \( d^\alpha \) in problem (2)

\[ d^P \parallel e_s^P, \quad d^S \perp e_s^S \]

Scattered wave of type \( \alpha \) (\( \alpha = P, S \))

\( U_0 \) ... displacement amplitude for wave of type \( \beta \) (\( \beta = P, S \)) in problem (1)
Ultrasonic Measurement Models

3-D scattering amplitude

\[ u^{\text{scatt}}(y, \omega) = U_0 \frac{A(e^\beta; e^P_s)}{r_s} \exp(ik_p r_s) + U_0 \frac{A(e^\beta; e^S_s)}{r_s} \exp(ik_s r_s) \]

However, it can be shown that

\[ A(e^\beta_i; e^\alpha_s) \cdot (-d^\alpha) = \int_{S_f} A(\tilde{v}_j^{(1)}, \tilde{\tau}_{ij}^{(1)}) dS \]

vector scattering amplitude of waves in problem (1) polarization vector of incident waves in problem (2)
The beam modeling terms and flaw scattering terms appear separately in this model but they still must be combined before the voltage can be obtained.
6) Now, assume a small 3-D flaw, i.e. where the incident waves do not vary significantly in amplitude over the surface of the flaw

\[
\hat{V}^{(1)}(x_1, x_2, x_3) = \hat{V}^{(1)}(0, 0, 0) = \hat{V}_0^{(1)}
\]
\[
\hat{V}^{(2)}(x_1, x_2, x_3) = \hat{V}^{(1)}(0, 0, 0) = \hat{V}_0^{(2)}
\]

Then

\[
V_R(\omega) = s(\omega) \left[ \frac{4\pi}{-ik_2 S_A} \frac{\rho_2 c_2}{\rho_1 c_1} \right] \left[ \hat{V}_0^{(1)} \hat{V}_0^{(2)} \right] \int_{S_f} A \left( \tilde{v}_j^{(1)}, \tilde{\tau}_{ij}^{(1)} \right) dS
\]

and, recall

\[
A \left( e_\beta^i ; e_\alpha^s \right) \cdot (-d_\alpha^i) = \int_{S_f} A \left( \tilde{v}_j^{(1)}, \tilde{\tau}_{ij}^{(1)} \right) dS
\]
Ultrasonic Measurement Models

Measurement Model (Thompson-Gray form)


\[ V_R(\omega) = s(\omega) \left[ \hat{V}_0^{(1)} \hat{V}_0^{(2)} \right] \left[ A(e_i^\beta ; e_s^\alpha) \cdot (-d^\alpha) \right] \left[ \begin{array}{c} \frac{4\pi}{-ik_2 S_A} \frac{\rho_2 c_2}{\rho_1 c_1} \end{array} \right] \]

beam models \[ A(e_i^\beta , e_s^\alpha , \omega) \]

plane wave far field scattering amplitude component

\[ t_A(\omega) = \left[ \hat{V}_0^{(1)} \hat{V}_0^{(2)} \right] \left[ A(e_i^\beta , e_s^\alpha , \omega) \right] \left[ \begin{array}{c} \frac{4\pi}{-ik_2 S_A} \frac{\rho_2 c_2}{\rho_1 c_1} \end{array} \right] \]
Ultrasonic Measurement Models

Measurement Model (Thompson-Gray form)

\[ V_R(\omega) = s(\omega) \left[ \hat{V}_0^{(1)} \hat{V}_0^{(2)} \right] \left[ A(\mathbf{e}_i^\beta, \mathbf{e}_s^\alpha, \omega) \right] \left[ \frac{4\pi}{-ik_2 S_A} \frac{\rho_2 c_2}{\rho_1 c_1} \right] \]

1) linear, reciprocal acoustic and elastic media (harmonic disturbances)

2) the output velocity for the transducer(s) can be written in a separable form

3) the pulser, receiver, transducers and cabling can be replaced by equivalent linear, time-shift invariant (LTI) systems.

4) the transducers are assumed to behave as piston radiators operating at high frequency

5) the incident waves are quasi-plane waves in the vicinity of the flaw

6) the scatterer is a small 3-D flaw where the incident waves do not vary significantly in amplitude over the surface of the flaw
6a) If instead, the scatterer is a small cylindrical flaw (e.g. side-drilled hole), with incident and scattered wave directions in the plane of the cross-section then we can assume

\[ \hat{V}^{(1)}(x_1, x_2, x_3) = \hat{V}^{(1)}(0, x_2, 0) \equiv \hat{V}_0^{(1)} \]

\[ \hat{V}^{(2)}(x_1, x_2, x_3) = \hat{V}^{(1)}(0, x_2, 0) \equiv \hat{V}_0^{(2)} \]

\[ A\left(\tilde{v}_j^{(1)}, \tilde{\tau}_{ij}^{(1)}, x\right) = A\left(\tilde{v}_j^{(1)}, \tilde{\tau}_{ij}^{(1)}, x_1, x_3\right) \quad (I, J = 1, 3) \quad \text{only} \]

In this case

\[ V_R(\omega) = S(\omega) \left[ \frac{4\pi}{-ik_2 S_A} \frac{\rho_2 c_2}{\rho_1 c_1} \right] \int_{\text{length}, L} \hat{V}_0^{(1)} \hat{V}_0^{(2)} dx_2 \int_{C_f} A\left(\tilde{v}_j^{(1)}, \tilde{\tau}_{ij}^{(1)}\right) ds \]
3-D scattering amplitude component of a 2-D, cylindrical geometry (end effects neglected) is just

$$A_{3D}\left(\mathbf{e}_i^\beta, \mathbf{e}_s^\alpha, \omega\right) = L \int_{C_f} A\left(\tilde{V}_j^{(1)}, \tilde{\tau}_{IJ}^{(1)}\right) ds$$

so that the measurement model can be expressed, finally, as

$$V_R\left(\omega\right) = s\left(\omega\right) \left[ \int_{length,L} \tilde{V}_0^{(1)}\tilde{V}_0^{(2)} d\chi_2 \right] \frac{A_{3D}\left(\mathbf{e}_i^\beta, \mathbf{e}_s^\alpha, \omega\right)}{L} \left[ \frac{4\pi}{-ik_2 S_A} \frac{\rho_2 c_2}{\rho_1 c_1} \right]$$

Relationship between 2-D and 3-D scattering amplitudes

\[ A_{2D}(e_i^\beta, e_s^\alpha, \omega) = \sqrt{\frac{2i\pi}{k}} \frac{A_{3D}(e_i^\beta, e_s^\alpha, \omega)}{L} \]

which gives the alternate form

\[ V_R(\omega) = s(\omega) \left[ \int_{\text{length},L} \hat{V}_0^{(1)}\hat{V}_0^{(2)} \, dx_2 \right] A_{2D}(e_i^\beta, e_s^\alpha, \omega) \left[ \sqrt{\frac{8\pi i}{k_2}} \frac{\rho_2c_2}{\rho_1c_1S_A} \right] \]
Measurement Model for 2-D scatterers

\[ V_R(\omega) = s(\omega) \left[ \int_{\text{length,} L} \hat{V}_0^{(1)} \hat{V}_0^{(2)} \, dx_2 \right] \frac{A_{3D} \left( e_i^\beta, e_s^\alpha, \omega \right)}{L} \left[ \frac{4\pi}{-i k_2 S_A} \frac{\rho_2 c_2}{\rho_1 c_1} \right] \]

1) linear, reciprocal acoustic and elastic media (harmonic disturbances)

2) the output velocity for the transducer(s) can be written in a separable form

3) the pulser, receiver, transducers and cabling can be replaced by equivalent linear, time-shift invariant (LTI) systems.

4) the transducers are assumed to behave as piston radiators operating at high frequency

5) the incident waves are quasi-plane waves in the vicinity of the flaw

6) the scatterer is cylindrical flaw with incident and scattered wave directions in the plane of the cross-section, where the incident waves do not vary significantly in amplitude over the small cross-section of the flaw
Ultrasonic Measurement Models

General Measurement Model (Auld form)

\[ V_R(\omega) = \frac{s(\omega)}{Z_r^A(\omega) V_A^{(1)}(\omega) V_B^{(2)}(\omega)} \int_{S_f} \left( t^{(1)} \cdot v^{(2)} - t^{(2)} \cdot v^{(1)} \right) dS \]

3-D small flaw (Thompson-Gray form)

\[ V_R(\omega) = s(\omega) \left[ \mathbf{\hat{V}}_0^{(1)} \mathbf{\hat{V}}_0^{(2)} \right] A(\mathbf{e}^B_i, \mathbf{e}^\alpha_s, \omega) \left[ \frac{4\pi}{-ik_2 S_A} \frac{\rho_2 c_2}{\rho_1 c_1} \right] \]

2-D small flaw (Schmerr-Sedov form)

\[ V_R(\omega) = s(\omega) \left[ \int_{\text{length},L} \mathbf{\hat{V}}_0^{(1)} \mathbf{\hat{V}}_0^{(2)} dx_2 \right] \frac{A_{3D}(\mathbf{e}^B_i, \mathbf{e}^\alpha_s, \omega)}{L} \left[ \frac{4\pi}{-ik_2 S_A} \frac{\rho_2 c_2}{\rho_1 c_1} \right] \]
Is the system linear and reciprocal?

Electrical linearity checks of pulser/receiver

For a transducer

\[ V_\infty I = V I_s \]

\[ I_0 = 0 \]

\[ I_s \]

\[ V_0 = 0 \]
Is the transducer acting as a piston?

Compare responses with theoretical predictions in a reference experiment, e.g.

\[ \text{Is a quasi-plane wave assumption valid?} \]

\[
\int_{S_f} \left( \frac{\tau_{ij}^{(1)}}{v_A^{(1)}} \frac{v_i^{(2)}}{v_B^{(2)}} - \frac{v_i^{(1)}}{v_A^{(1)}} \frac{\tau_{ij}^{(2)}}{v_B^{(2)}} \right) n_j \, dS
\]

as computed with a non-paraxial beam model

with

\[
\left[ \frac{4\pi\rho_2 c_2}{-ik_2} \right] \int_{S_f} \hat{V}^{(1)} \hat{V}^{(2)} a(\hat{v}_j^{(1)}, \hat{\tau}_{ij}^{(1)}) \, dS
\]
Ultrasonic Measurement Models

Is a small flaw assumption valid?

Compare
\[ \int_{S_f} \hat{V}^{(1)} \hat{V}^{(2)} a \left( \tilde{v}_j^{(1)}, \tilde{\tau}_{ij}^{(1)} \right) dS \]

with
\[ \left[ \hat{V}_0^{(1)} \hat{V}_0^{(2)} \right] A_{3D} \left( e_i^\beta, e_s^\alpha, \omega \right) \] (3-D case)

or
\[ \left[ \frac{1}{L_{\text{length}, L}} \int \hat{V}_0^{(1)} \hat{V}_0^{(2)} dx_2 \right] A_{3D} \left( e_i^\beta, e_s^\alpha, \omega \right) \] (cylindrical case)
What supporting measurements are needed?

**System factor**

Obtain in separate experiment

**Velocity, Attenuation**

Obtain in separate experiments

**Transducer effective radius, focal length**
Ultrasonic Measurement Models

Summary

A measurement model form is available suitable for simulating most NDE inspections.

To make these models useful one needs to have:
- ultrasonic beam models
- flaw scattering models
- experimental determination of essential parameters

Once all these elements are available one has a very powerful tool for designing tests, optimizing components, and replacing expensive experimental procedures.