Ultrasonic System Measurement Model - I
Learning Objectives

Use of a simple fluid model to obtain a complete measurement model

extension to a fluid-solid model
Ultrasonic System Model – Single Fluid
Consider the sound generation process:

\[ F_t(\omega) = \rho c S_t v_0(\omega) = t_G(\omega) V_i(\omega) \]

\( S_t \) ... area of the transmitting transducer

\( t_G(\omega) \) ... transfer function for pulser, cabling, transducer
In the paraxial approximation, the beam produces a quasi-plane wave incident on the flaw given by:

$$p(x_s, \omega) = P_{inc} \exp(ik e_{i0} \cdot x_s)$$
Ultrasonic System Model – Single Fluid

\[ p(x_s, \omega) = P_{inc} \exp(ik e_{i0} \cdot x_s) \]

\[ P_{inc} = \rho cv_0(\omega) C_t(\omega) \exp(ikz_i) \]

\[ = \frac{t_G(\omega)V_i(\omega)}{S_t} C_t(\omega) \exp(ikz_i) \]

Note: since

\[ p(x_s, \omega) = \frac{-i\omega \rho v_0}{2\pi} \int_{S_t} \frac{\exp(ikr_i)}{r_i} dS = \rho cv_0 C(\omega) \exp(ikz_i) \]

\[ C_t(\omega) = \frac{-ik}{2\pi} \exp(-ikz_i) \int_{S_t} \frac{\exp(ikr_i)}{r_i} dS \]
To calculate the diffraction coefficient, however, we can use a multi-Gaussian beam model:

\[
C_t(\omega) = \frac{-ik}{2\pi} \exp(-ikz_i) \int_{s_t} \frac{\exp(ikr_i)}{r_i} \, dS
\]

\[
p = \rho c v_0 \exp[ikz_i] \sum_{n=1}^{10} \frac{A_n}{q_1(0)_n + z_i} \exp\left[\frac{ik\rho_i^2}{2} \frac{1}{q_1(0)_n + z_i}\right] \\
= \rho c_p \nu_0 \exp[ik_p z_i] C_t(\omega)
\]

\[
C_t(\omega) = \sum_{n=1}^{10} \frac{A_n}{q_1(0)_n + z_i} \exp\left[\frac{ik\rho_i^2}{2} \frac{1}{q_1(0)_n + z_i}\right] \\
\left[q_1(0)_n\right] = \frac{-ika^2 / 2}{B_n}
\]
For the waves scattered from the flaw

\[ p^{\text{scatt}}(\mathbf{x}, \omega) = P_{\text{inc}} \frac{A(\mathbf{e}_{i0}; \mathbf{e}_s)}{r_s} \exp(ikr_s) \]

\( A(\mathbf{e}_{i0}; \mathbf{e}_s) \) plane wave far-field scattering amplitude
Now, consider the scattered waves at the receiving transducer:

\[
F_B(\omega) = 2 \int_{S_r} p^{\text{scatt}} dS = 2P_{inc} \int_{S_r} A(e_{io}, e_s) \frac{\exp(ikr_s)}{r_s} dS
\]
In the paraxial approximation (on reception) \[ A(e_{io}, e_s) \approx A(e_{io}, e_{so}) \]

and we have \[ F_B(\omega) = 2P_{inc} A(e_{io}, e_{so}) \int_{S_r} \frac{\exp(ikr_s)}{r_s} dS \]
If receiving transducer were acting as a transmitter it would produce a pressure field given by

\[ p(x_s, \omega) = \frac{-i \omega \rho v_0}{2\pi} \int_{S_r} \frac{\exp(ikr_s)}{r_s} dS = \rho cv_0 C_r(\omega) \exp(ikz_s) \]
Thus, \[ \int_{S_r} \frac{\exp(ikr_s)}{r_s} dS = \frac{2\pi}{-ik} C_r(\omega) \exp(ikz_s) \]

and \[ F_B(\omega) = 2P_{inc} A(e_{io}, e_{so}) \int_{S_r} \frac{\exp(ikr_s)}{r_s} dS \]

becomes \[ = P_{inc} A(e_{io}, e_{so}) C_r(\omega) \left[ \frac{4\pi}{-ik} \right] \exp(ikz_s) \]
Again, we could compute this diffraction correction with our multi-Gaussian beam model

$$C_r(\omega) = \frac{-ik}{2\pi} \exp(-ikz_s) \int_{s_r} \frac{\exp(ikr_s)}{r_s} dS$$

$$C_t(\omega) = \sum_{n=1}^{10} \left[ \frac{A_n}{q_1(0)_n + z_s} \right] \exp \left[ \frac{ik\rho_s^2}{2} \left( \frac{1}{q_1(0)_n + z_s} \right) \right]$$
Ultrasonic System Model – Single Fluid

Sound reception process:

\[ V_R(\omega) = t_R(\omega) F_B(\omega) \]

\[ t_R(\omega) \] is the transfer function for the transducer, cabling, and receiver.
Combining all these relations, we find:

\[
P_{inc} = \frac{t_G(\omega) V_i(\omega)}{S_t} C_t(\omega) \exp(i k z_i)
\]

\[
F_B(\omega) = P_{inc} A(e_{io}, e_{so}) C_r(\omega) \left[ \frac{4\pi}{-ik} \right] \exp(i k z_s)
\]

\[
V_R(\omega) = t_R(\omega) F_B(\omega)
\]
Ultrasonic System Model – Single Fluid

\[ V_R(\omega) = t_G(\omega)t_R(\omega)V_i(\omega)e^{(ikz_i + ikz_s)}C_l(\omega)A(e_{io}, e_{so})C_r(\omega) \left[ \frac{4\pi}{-ikS_i} \right] \]
\[ V_R(\omega) = s(\omega) C_t(\omega) \exp(ikz_i) C_r(\omega) \exp(ikz_s) A(e_{io}, e_{so}) \left[ \frac{4\pi}{-ikS_t} \right] \]

\( s(\omega) \) … system function (pulser/receiver, cabling, transducer)
Ultrasonic System Model-Single Fluid

\[ V_R(\omega) = s(\omega)C_t(\omega)\exp(ikz_i)C_r(\omega)\exp(ikz_s)A(e_{io}; e_{so}, \omega) \left[ \frac{4\pi}{-ikS_t} \right] \]

received voltage

pulser, receiver
cabling, transducers

beam propagation and
diffraction effects going
from the transmitting
transducer to the flaw
and from the flaw to the
receiving transducer

flaw
scattering
area of
transmitter

When attenuation of the fluid is important, we need to also
include terms \( \exp[-\alpha(\omega)z_i - \alpha(\omega)z_s] \)

\( \alpha(\omega) \) … frequency dependent attenuation
For pitch-catch

\[
V_R(\omega) = s(\omega) \hat{V}_t(\omega) \hat{V}_r(\omega) A(e_{io}; e_{so}, \omega) \left[ \frac{4\pi}{-ikS_t} \right]
\]

\[
\hat{V}_t(\omega) = C_t(\omega) \exp(-\alpha z_i) \exp(ikz_i)
\]

\[
\hat{V}_r(\omega) = C_r(\omega) \exp(-\alpha z_s) \exp(ikz_s)
\]

\[
\hat{V}_t(\omega) = \hat{V}_r(\omega) = \hat{V}(\omega)
\]

For pulse-echo

\[
V_R(\omega) = s(\omega) \left[ \hat{V}(\omega) \right]^2 A(e_{io}; e_{so}, \omega) \left[ \frac{4\pi}{-ikS_t} \right]
\]
Thompson-Gray Measurement Model – Fluid-Solid (Pulse-Echo)

\[ V_R(\omega) = s(\omega) \left[ \hat{V}(\omega) \right]^2 A(\omega) \left[ \frac{4\pi}{-ik r_2 S_A} \frac{\rho_2 c_{r_2}}{\rho_1 c_{p_1}} \right] \]

\[ A(\omega) = \left[ \mathbf{A} \left( \mathbf{e}_{io}^r, -\mathbf{e}_{io}^r \right) \cdot \left( -\mathbf{d}_0^r \right) \right] \quad (r = p \text{ or } s) \]
Thompson-Gray Measurement Model – Fluid-Solid (Pulse-Echo)

\[ V_R(\omega) = s(\omega) \left[ \hat{V}(\omega) \right]^2 A(\omega) \left[ \frac{4\pi}{-ik_{r2}S_A} \frac{\rho_2c_{r2}}{\rho_1c_{p1}} \right] \]

\( (r = p \text{ or } s) \)

\[ \hat{V}(\omega) = P(\omega)M(\omega)T(\omega)C(\omega) \]

\[ P(\omega) = \exp(ik_{p1}z_1 + ik_{r2}z_2) \quad \text{... propagation} \]

\[ M(\omega) = \exp(-\alpha_{p1}z_1 - \alpha_{r}z_2) \quad \text{... attenuation} \]

\[ T(\omega) = T_{12} \quad \text{... plane wave transmission coefficient (velocity/velocity)} \]

\[ C(\omega) \quad \text{... diffraction coefficient} \]