Impedance Concepts and Thévenin Equivalent Systems
Learning Objectives

Electrical impedance

Description of 1-D waves in a fluid
  equation of motion/constitutive equation
  plane waves - pulses and harmonic waves
  frequency and wavelength
  acoustic impedance

Thévenin equivalent electrical circuits
Concept of Impedance

basic circuit elements

$$V(t) = R \cdot I(t)$$

$$C \frac{dV(t)}{dt} = I(t)$$

$$V(t) = L \frac{dI(t)}{dt}$$
For alternating (harmonic) voltages and currents

\[ I(t) = I_0 \exp(-i\omega t) \]

\[ V(t) = V_0 \exp(-i\omega t) \]

Resistor:
\[ V_0 = R I_0 \]

Capacitor:
\[ V_0 = \frac{I_0}{-i\omega C} \]

Inductor:
\[ V_0 = -i\omega L I_0 \]

general impedance:
\[ V_0 = Z(\omega) I_0 \]

Electrical impedance \( Z = Z^e \) (ohms)
1-D plane wave traveling wave in a fluid

\[ p(x, t) \] \quad \text{... pressure in the fluid}
\[ \rho \] \quad \text{... fluid density}
\[ v_x (x, t) \] \quad \text{... velocity}

\[ \sum \mathbf{F} = m \mathbf{a} \]

\[ -\left( p + \frac{\partial p}{\partial x} dx \right) dydz + pdydz = \rho dx dy dz \frac{\partial v_x}{\partial t} \]

Equation of motion

\[ -\frac{\partial p}{\partial x} = \rho \frac{\partial v_x}{\partial t} \]
Constitutive equation

\[ p = -K \frac{\partial u_x}{\partial x} \]

K \quad \ldots \text{compressibility of the fluid}

\[ u_x \quad \ldots \text{displacement} \]

\[ \frac{\partial u_x}{\partial x} = \frac{\Delta V}{V} \quad = \text{relative change in volume} \]

\[ -\frac{\partial^2 p}{\partial x^2} = \rho \frac{\partial^2 (\partial u_x / \partial x)}{\partial t^2} = \frac{\rho}{K} \frac{\partial^2 p}{\partial t^2} \]

1-D wave equation

\[ \frac{\partial^2 p}{\partial x^2} - \frac{1}{c_f^2} \frac{\partial^2 p}{\partial t^2} = 0 \quad c_f = \sqrt{\frac{K}{\rho}} \quad \text{wave speed in the fluid} \]
1-D plane wave solutions

\[ p = f(t - x/c_f) + g(t + x/c_f) \]

Harmonic waves (or Fourier transforms of transient waves)

\[ p = A(f) \exp[-2\pi if(t - x/c_f)] + B(f) \exp[-2\pi if(t + x/c_f)] \]
Fourier transform pair

\[ V(f) = \int_{-\infty}^{+\infty} v(t) \exp(2\pi if t) dt \]

\[ v(t) = \int_{-\infty}^{+\infty} V(f) \exp(-2\pi if t) df \]

Let \[ V(f) = \tilde{V}(f) \exp\left[ 2\pi if \left( \frac{x}{c_f} \right) \right] \]

then \[ v(t) = \int_{-\infty}^{+\infty} \tilde{V}(f) \exp\left[ 2\pi if \left( \frac{x}{c_f} - t \right) \right] df \]

1-D plane wave traveling in the +x-direction

Thus, \( v(t) \) is a superposition of plane waves
Now, let \( u = (t - x / c_f) \)

Then
\[
\int_{-\infty}^{+\infty} \tilde{V}(f) \exp(-2\pi ifu) df = \tilde{v}(u) = \tilde{v}(t - x / c_f)
\]

where \( \tilde{v}(t) \leftrightarrow \tilde{V}(f) \)

so
\[
\tilde{v}(t - x / c_f) = \int_{-\infty}^{+\infty} \tilde{V}(f) \exp[2\pi if(x / c_f - t)] df
\]

Thus, we can always synthesize a traveling plane wave pulse with a superposition of harmonic plane waves.
various forms of a harmonic plane wave traveling in the $+x$-direction

\[
p = A \exp\left(2\pi ifx / c_f - 2\pi ift\right) = A \exp\left(2\pi ifx / c_f - i\omega t\right) = A \exp\left(ikx - i\omega t\right) = A \exp\left(2\pi ix / \lambda - i\omega t\right)
\]

\[
f \quad \text{... frequency (cycles/sec)}
\]

\[
\omega \quad \text{...frequency (rad/sec)}
\]

\[
k \quad \text{... wave number (rad/length)}
\]

\[
\lambda \quad \text{...wavelength (length/cycle)}
\]
$p = A \exp \left( 2\pi i f x / c_f - 2\pi i f t \right)$

magnitude of the pressure at a fixed $x$ versus time, $t$:

$$2\pi i f T = 2\pi i$$

$\omega = 2\pi f$  circular frequency (rad/sec)

rad/cycle
\[ p = A \exp\left(2\pi ifx / c_f - 2\pi ift\right) \]

magnitude of the pressure at a fixed time, \( t \), versus distance, \( x \):

\[ \lambda \ldots \text{wavelength (length/cycle)} \]

\[ f_x = \frac{1}{\lambda} \quad \text{spatial frequency (cycles/length)} \]

\[ k = 2\pi f_x = \frac{2\pi}{\lambda} \quad \text{spatial circular frequency (wave number) (rad/length)} \]
\[ p = A \exp\left( \frac{2\pi ifx}{c_f} - 2\pi ift \right) \]

magnitude of the pressure at a fixed time, \( t \), versus distance, \( x \):

\[ \lambda \ldots \text{wavelength (length/cycle)} \]

\[ 2\pi f \lambda / c_f = 2\pi i \]

\[ \rightarrow f\lambda = c_f \]

fundamental relationship that shows how the frequency of a plane harmonic traveling wave is related to its wavelength
Example:

Consider a 5MHz plane wave traveling in water (c = 1500m/sec). What is the wavelength?

\[
\lambda = \frac{1.5 \times 10^6 \text{ mm/ sec}}{5 \times 10^6 \text{ cycle/ sec}} = 0.3 \text{ mm/ cycle}
\]
For a plane pressure wave traveling in the x-direction

\[ p = f \left( t - x / c_f \right) \]

Let \( u = t - x / c_f \)

\[ \frac{\partial p}{\partial x} = \frac{df(u)}{du} \frac{\partial u}{\partial x} = -\frac{1}{c_f} f' \]

\[ -\frac{\partial p}{\partial x} = \rho \frac{\partial v_x}{\partial t} \]

\[ \frac{\partial v_x}{\partial t} = \frac{1}{\rho c_f} f' \]

\[ v_x = \frac{1}{\rho c_f} \int f' \, dt = \frac{1}{\rho c_f} \int \frac{df}{du} \, du = \frac{f}{\rho c_f} = \frac{p}{\rho c_f} \]
\[ p = \rho c_f v_x \]

Specific acoustic impedance of a plane wave (pressure/velocity)

\[ z^a = \rho c_f \]

For more general (harmonic) waves \( F = Z^a (\omega) v \)
Thévenin equivalent circuit (harmonic voltages and currents)
Determining the Thévenin equivalent source

\[ V_s(\omega) \quad Z_{eq}(\omega) \quad V_0 \quad \text{open-circuit voltage} \]

\[ V_s = V_0 \]

Determining the Thévenin equivalent impedance

\[ V_s - V_L = Z_{eq}^e I \]

\[ V_L = R_L I \]

\[ Z_{eq} = R_L \left( \frac{V_s}{V_L} - 1 \right) \]
Example: find the Thévenin equivalent circuit for the following circuit
\[ V_i - V_0 = IR \]

\[ V_0 = \frac{I}{-i\omega C} \]

eliminating I, we find

\[ V_0 = \frac{V_i}{1 - i\omega RC} \quad \text{Thévenin equivalent source} \]
\[ V_i - V_L = I_1 R \quad V_L = I_2 R_L \quad V_L = \frac{(I_1 - I_2)}{-i\omega C} \]

Eliminating \( I_1 \), \( I_2 \) gives

\[ V_L = \frac{V_i}{(1 - i\omega RC) + R / R_L} \]

So

\[ Z_{eq} = R_L \left( \frac{V_0}{V_L} - 1 \right) = R_L \left\{ \frac{(1 - i\omega RC) + R / R_L}{(1 - i\omega RC)} - 1 \right\} \]

\[ = R_L \left\{ \frac{R / R_L}{(1 - i\omega RC)} \right\} = \frac{R}{(1 - i\omega RC)} \]
\[ V_s(\omega) = \frac{V_i(\omega)}{1 - i\omega RC} \quad Z_{eq}(\omega) = \frac{R}{1 - i\omega RC} \]
In many EE books the equivalent impedance is obtained by simply shorting out (removing) the source and examining the ratio $V/I$ at the output port:

$$V = R(I - I_1)$$
$$V = \frac{I_1}{-i\omega C}$$

$$V = R(I + i\omega CV)$$
$$V(1 - i\omega RC) = RI$$

$$Z_{eq} = \frac{V}{I} = \frac{R}{1 - i\omega RC}$$