Flaw Scattering Models
Learning Objectives

Far-field scattering amplitude

Kirchhoff approximation
Born approximation
Separation of Variables

Examples of scattering of simple shapes
(spherical pore, flat crack, side-drilled hole)
At many wavelengths away from the flaw the scattered waves are spherical waves

\[ p^{\text{scatt}}(\mathbf{y}, \omega) = p_0 A(e_i; e_s) \frac{\exp(ikr_s)}{r_s} \]

\( A \) is called the plane wave far-field scattering amplitude.
Flaw Scattering

Incident plane wave $p_0$... pressure amplitude

Flaw Model

Fluid Model

plane wave far field scattering amplitude of the flaw

$$A(e_i; e_s) = \frac{-1}{4\pi} \int_{S_f} \left[ \frac{\partial \tilde{p}}{\partial n} + ik (e_s \cdot n) \tilde{p} \right] \exp(-ik x_s \cdot e_s) dS(x_s)$$

$$\tilde{p} = \frac{p(x_s, \omega)}{p_0}$$
Flaw Scattering

\[ A(e_i; e_s) = \frac{-1}{4\pi} \int_{S_f} \left[ \frac{\partial \tilde{p}}{\partial n} + ik (e_s \cdot n) \tilde{p} \right] \exp(-ikx_s \cdot e_s) dS(x_s) \]

\[ \tilde{p} = \frac{p(x_s, \omega)}{p_0} \]

To obtain the far field scattering amplitude, we need to know the pressure and velocity on the surface of the flaw due to the incident and scattered waves.

These quantities can be found by solving a flaw scattering boundary value problem.
Flaw Scattering

Elastic Solid

\[ u^{\text{scatt}}(y_\omega) = U_0 \frac{A(e^\beta_i ; e^P_s)}{r_s} \exp(ik_p r_s) + U_0 \frac{A(e^\beta_i ; e^S_s)}{r_s} \exp(ik_s r_s) \]

scattered displacement from a flaw

- \( \alpha \) wave
- \( \beta \) wave

\[ U_0 \ldots \text{displacement amplitude for wave of type } \beta (\beta = P, S) \text{ in problem (1)} \]

Scattered wave of type \( \alpha (\alpha = P, S) \)
Flaw Scattering

\[ A \left( e_i^\beta ; e_s^\alpha \right) \]

Vector scattering amplitude for a scattered wave of type \( \alpha \) due to a plane wave of unit displacement amplitude and type \( \beta \)

\[
A \left( e_i^\beta ; e_s^P \right) = \left( f^{P;\beta} \cdot e_s^P \right) e_s^P \quad (\beta = P, S)
\]

\[
A \left( e_i^\beta ; e_s^S \right) = \left[ f^{S;\beta} - \left( f^{S;\beta} \cdot e_s^S \right) e_s^S \right]
\]

elastic constants

\[
f^{\alpha;\beta} = f_l^{\alpha;\beta} e_l = - \frac{C_{lkpj}}{4\pi \rho c_\alpha^2} \int_s \left[ \left( \frac{\partial \tilde{u}_p}{\partial x_j} \right) n_k + ik_\alpha e_{sk}^\alpha n_p \tilde{u}_j \right] \exp \left( -ik_\alpha x_s \cdot e_s^\alpha \right) dS(x_s)
\]

displacements and displacement gradients

\[
(\alpha = P, S) \quad (\beta = P, S)
\]
Flaw Scattering

3-D scattering amplitude

\[ u_{\text{scatt}}(y, \omega) = U_0 \frac{A(e_i^\beta; e_s^p)}{r_s} \exp(ik_p r_s) + U_0 \frac{A(e_i^\beta; e_s^s)}{r_s} \exp(ik_s r_s) \]

The received voltage in some measurement models is related to a specific component of the vector scattering amplitude given by

\[ A(\omega) = A(e_i^\beta; e_s^\alpha) = A(e_i^\beta; e_s^\alpha) \cdot (-d^\alpha) \]

polarization vector of an incident wave from receiving transducer (to be discussed shortly)
Flaw Scattering

\[ A(\omega) \] can be obtained from:

Analytical/Numerical methods
- Separation of variables
- Boundary Elements (BEM)
- Finite Elements (FEM)
- Method of Optimal Truncation (MOOT)
- T-matrix
- Kirchhoff Approximation
- Born Approximation

or \( A(\omega) \) can be found experimentally through deconvolution:

\[ V_R(\omega) = E(\omega)A(\omega) \]

\[ A(\omega) = \frac{V_R(\omega)E^*(\omega)}{|E(\omega)|^2 + n^2} \]

measure model, measure
Kirchhoff Approximation for a Volumetric Flaw

on the "lit" surface: fields are those from plane waves interacting with a plane surface

on the rest of the surface the fields are assumed to be zero
It has been recently shown that the pulse-echo scattering amplitude response, \( A(\omega) \), for a stress-free flaw (void or crack) in an elastic solid is identical to the scalar (fluid) scattering amplitude:

\[
A(\omega) = A\left(\mathbf{e}_i^\beta; -\mathbf{e}_i^\beta\right) = \frac{-ik_\beta}{2\pi} \int_{S_{\text{lit}}} \left( \mathbf{e}_i^\beta \cdot \mathbf{n} \right) \exp\left(2ik_\beta \mathbf{x}_s \cdot \mathbf{e}_i^\beta\right) dS(\mathbf{x}_s)
\]
For the pulse-echo response of a spherical void of radius $a$

$$A(\omega) \equiv A(e_i; -e_i, \omega) = \frac{-a}{2} \exp(-ika) \left[ \exp(-ika) - \frac{\sin(ka)}{ka} \right]$$

$a = 1 \text{ mm}$
$c = 5900 \text{ m/s}$
Comparison with Kirchhoff solution with exact separation of variables solution for the P-P scattering amplitude of a void in an elastic solid (pulse-echo)

leading edge response

"creeping" wave

Flaw Scattering-Spherical Void
Inverting the scattering amplitude (fluid model) into the time domain

\[ A(t) \equiv A(e_i; -e_i, t) = \frac{-a}{2} \left[ \delta\left( t + \frac{2a}{c} \right) - \frac{c}{2a} U\left( \frac{-2a}{c}, 0; t \right) \right] \]

\[ U(t_1, t_2; t) = \begin{cases} 
1 & \text{if } t_1 < t < t_2 \\
0 & \text{otherwise}
\end{cases} \]
Exact solution for a void in an elastic solid (time domain) vs the Kirchhoff approximation solution.
Although the Kirchhoff approximation is formally a high frequency approximation (\(ka >> 1\)), numerical tests have shown it capable of producing good agreement (<1 dB difference) for the peak-to-peak time domain response of the spherical void down to \(ka = 1\) provided the bandwidth is sufficient.

- white: differences <1 dB
- gray: >1 dB but < 1.5 dB
- black: > 1.5 dB
Flaw Scattering-Inclusion

Leading Edge Response - General Convex Inclusion
(scattered mode same as incident mode)

High frequency, stationary phase approximation

\[ \theta_i = \alpha \]

\[ A(e_i, e_s, \omega) = R_{12} \frac{\sqrt{R_1 R_2}}{2} \exp \left[ -ik |e_i - e_s| d \right] \]

Plane wave reflection coefficient
Gaussian curvature of flaw surface at stationary phase point
Flaw Scattering-Inclusion

Leading Edge Response - General Convex Inclusion
(scattered mode same as incident mode)

In the time domain

\[ A(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\omega) \exp(-i\omega t) d\omega \]

\[ A(\mathbf{e}_i; \mathbf{e}_s, t) = R_{12} \sqrt{R_1 R_2} \delta(t + |\mathbf{e}_i - \mathbf{e}_s| d / c) \]
Flaw Scattering-Crack

Kirchhoff Approximation – Flat Elliptical Cracks

\[ A(e_i; e_s) = \frac{-ia_1a_2(e_i \cdot n)}{|e_i - e_s| r_e} J_1(k|e_i - e_s|r_e) \]

\[ r_e = \sqrt{a_1^2 (e_q \cdot u_1)^2 + a_2^2 (e_q \cdot u_2)^2} \]

plane normal to \( e_q \)

\[ e_q = \frac{e_i - e_s}{|e_i - e_s|} \]
General pitch-catch crack response-circular crack

The magnitude of the P-wave pulse-echo far-field scattering amplitude versus frequency for a 1 mm radius circular crack in steel with an angle of incidence of \( \theta \) from the crack normal.
Comparison of the Kirchoff approximation and MOOT for a 0.381 mm radius circular crack at an incident angle of 45 degrees (pulse-echo)
When $\mathbf{e}_q$ is parallel to the crack normal, $\mathbf{n}$:

$$A(\mathbf{e}_i;\mathbf{e}_s) = \frac{-ik(\mathbf{e}_i \cdot \mathbf{n})a_1a_2}{2}$$

Special case - pulse echo:

$$A(\mathbf{e}_i;-\mathbf{e}_i) = \frac{ika_1a_2}{2}$$

The magnitude of the P-wave pulse-echo far-field scattering amplitude versus frequency for a 1 mm radius circular crack in steel at normal incidence.
Comparison of the Kirchoff approximation and MOOT for a 0.381 mm radius crack at normal incidence (pulse-echo)

Flaw Scattering-Crack
Flaw Scattering-Crack

Inverting these results into the time domain

$$\mathbf{e}_i \cdot \mathbf{n} \neq 0$$

$$A(t) = A(\mathbf{e}_i; \mathbf{e}_s, t) = \begin{cases} 
- \frac{a_1 a_2 c (\mathbf{e}_i \cdot \mathbf{n})}{\pi |\mathbf{e}_i - \mathbf{e}_s|^2 r_e^2 \sqrt{(|\mathbf{e}_i - \mathbf{e}_s| r_e / c)^2 - t^2}} & t < |\mathbf{e}_i - \mathbf{e}_s| r_e / c \\
0 & \text{otherwise}
\end{cases}$$

Example: pulse-echo

flash points
At normal incidence, for pulse-echo

\[ A(e_i; -e_i, t) = \frac{-a_1 a_2}{2c} \frac{d\delta(t)}{dt} \]

\[ A(e_i; e_s) \]
A planar crack is a very “specular” reflector
Although the Kirchhoff approximation is formally a high frequency approximation (\( ka >>1 \)), numerical tests have shown it capable of producing good agreement (<1 dB difference) for the peak-to-peak time domain response of a circular crack at normal incidence down to \( ka =1.5 \) provided the bandwidth is sufficient.

white: differences <1 dB
gray: >1 dB but < 1.5 dB
black: > 1.5 dB
Comparison of synthesized waveforms scattered from a 0.381 mm radius crack using the Kirchhoff approximation and MOOT using frequencies from 0-25 MHz approximately (pulse echo)
0-15 degrees from crack normal
Flaw Scattering-Crack

20-35 degrees from crack normal
Flaw Scattering-Crack

40-55 degrees from crack normal
60-70 degrees from crack normal
Flaw Scattering-Crack

75-85 degrees from crack normal
Comparison of peak-to-peak values of MOOT and Kirchhoff versus angle of incidence for the 0.381mm radius crack (pulse-echo)
The arrow shows where agreement is within 1 dB
Now consider the scattering amplitude with narrow band Gaussian window. The central frequency is 10 MHz, and the bandwidth is 1 MHz. Radius of the crack $a = 0.381\text{mm}$.
the range of angles where the agreement is good is significantly reduced
Flaw Scattering-Crack

Now consider the scattering amplitude with wider band Gaussian window
The central frequency is 10 MHz, and the bandwidth is 6 MHz
Radius of the crack \( a = 0.381 \text{mm} \)
The range of excellent agreement now is back to angles as great as 45 degrees or more.
Flaw Scattering-Crack

Maximum incident angle where the peak-to-peak time domain agreement between the Kirchhoff approximation and the exact solution for a circular crack is less than 1 dB for $ka = 5.0$ (other $ka$ values shown same trend).
Kirchhoff approximation for pulse-echo scattering of a side-drilled hole (incident direction in plane perpendicular to the hole axis)

\[
A\left(e_i^\beta ; -e_i^\beta \right) = \frac{(k_\beta b)L}{2} \left[ J_1 \left( 2k_\beta b \right) - iS_1 \left( 2k_\beta b \right) \right] + \frac{i(k_\beta b)L}{\pi}
\]

Bessel function \hspace{1cm} Struve function
Comparison with exact 2-D separation of variables solution for P-waves (pulse-echo)

The three-dimensional normalized pulse-echo P-wave scattering amplitude versus normalized wave number for a side drilled hole in the Kirchhoff approximation (solid line) and from the exact two-dimensional separation of variables solution (dashed line).
Comparison with exact 2-D separation of variables solution for S-waves (pulse-echo_)

The three-dimensional normalized pulse-echo SV-wave scattering amplitude versus normalized wave number for a side drilled hole in the Kirchhoff approximation (solid line) and from the exact two-dimensional separation of variables solution (dashed line).
Flaw Scattering

Kirchhoff approximation - Summary

For volumetric flaws - the Kirchhoff approximation properly models the leading edge signal as long as \( ka > 1 \) approximately but does not model other waves (creeping waves, etc.)

For cracks – the Kirchhoff approximation models the flash point signals properly in pulse-echo as long as \( ka > 1 \) approximately and the incident angle is less than about 50 degrees for wide band responses. For narrow band responses this angle is considerably reduced to as little as 15-20 degrees.
The Born Approximation

The Born approximation assumes that the material of an inclusion differs little from the host material so that to first order the incident wave passes through the inclusion unchanged (weak scattering, low frequency approximation)
Flaw Scattering - Inclusion

The Born approximation generally is developed from a volume integral expression for the scattering amplitude

\[ A(e_i^B; e_s^a) = \frac{-d_q^a}{4\pi\rho c_\alpha^2} \int V_f \left[ \Delta \rho \omega^2 \tilde{u}_q + i k_{e_{sk}}^\alpha \Delta C_{k_{qm},j} \frac{\partial \tilde{u}_m}{\partial x_j} \right] \exp(-i k_{e_s} x \cdot e_s^a) dV(x) \]

fields in the flaw replaced by incident fields

\[ \tilde{u}_q = \tilde{u}_q^{\text{incident}} \]

\[ \frac{\partial \tilde{u}_m}{\partial x_j} = \frac{\partial \tilde{u}_m^{\text{incident}}}{\partial x_j} \]
For a spherical inclusion in pulse-echo

\[ A(e_i^\beta; -e_i^\beta) = -4k_\beta^2 b^3 F \frac{j_1(2k_\beta b)}{2k_\beta b} \]

where

\[ F = \frac{1}{2} \left( \frac{\Delta \rho}{\rho} + \frac{\Delta c}{c_\beta} \right) \]

relative density difference  relative wave speed difference
Corresponding time domain impulse response for a spherical inclusion (pulse-echo)

Flaw Scattering -Inclusion

- Front surface response: $\frac{-2b}{c_\alpha}$
- Back surface response: $\frac{2b}{c_\alpha}$

Diagram showing the impulse response with time axes and a spherical representation of the flaw.
Comparison with "exact" separation of variables solution for the pulse-echo response of a spherical inclusion by synthesizing a time-domain response from frequencies ranging from 0-20 MHz approximately for a 1 mm radius inclusion (10 % differences in properties)

The time domain pulse-echo P-wave response of a 1 mm radius spherical inclusion in steel where the density and compressional wave speed are both ten percent higher than the host steel. Solid line: Born approximation, dashed line: separation of variables solution.
Comparison with "exact" separation of variables solution for the pulse-echo response of a spherical inclusion by synthesizing a time-domain response from frequencies ranging from 0-20 MHz approximately for a 1 mm radius inclusion (50% differences in properties)

The time domain pulse-echo P-wave response of a 1 mm radius spherical inclusion in steel where the density and compressional wave speed are both fifty percent higher than the host steel. Solid line: Born approximation, dashed line: separation of variables solution.
Doubly Distorted Born Approximation

\[
A_{\text{Born}}(\mathbf{e}_i^\beta; -\mathbf{e}_i^\beta) = -4k_\beta b^3 F \frac{j_1(2k_\beta b)}{2k_\beta b} \\
F = \frac{1}{2} \left( \frac{\Delta \rho}{\rho} + \frac{\Delta c}{c_\beta} \right)
\]

replace wave speeds of host material by that of the flaw in the Born approximation

for the pulse-echo response of a spherical inclusion

\[
A(\mathbf{e}_i^\beta; -\mathbf{e}_i^\beta)_{\text{DDBA}} = -4k_{f\beta}^2 b^3 \tilde{F} \frac{j_1(2k_{f\beta} b)}{2k_{f\beta} b}
\]

where

\[
\tilde{F} = \frac{1}{2} \left( \frac{\Delta \rho}{\rho_f} + \frac{\Delta c}{c_{f\beta}} \right)
\]
The time domain pulse-echo P-wave response of a 1 mm radius spherical inclusion in steel where the density and compressional wave speed are both fifty percent higher than the host steel. Solid line: Doubly Distorted Born approximation, dashed line: separation of variables solution.

Flaw Scattering - Inclusion

Front surface amplitude response is improved, and relative time of arrival of back surface response is correct, but there is an error in absolute time of arrivals (50% differences in properties)
Why is the front surface response amplitude improved by the Doubly Distorted Born Approximation?

Answer: because

\[
\frac{\Delta \rho}{\rho} = \frac{\Delta c}{c}
\]

\(\tilde{F} \approx R_{12}^{\beta;\beta}\) (plane wave reflection coefficient)
This suggests we apply a phase correction to the doubly distorted Born approximation and replace $\tilde{F}$ by $R_{12}^{\beta;\beta}$ resulting in the modified Born approximation (MBA) for the pulse-echo response of a spherical inclusion

$$A\left(e_i^\beta; -e_i^\beta\right)_{MD^2} = -4k_{f\beta}^2 b^3 R_{12}^{\beta;\beta} \exp\left[2ik_{f\beta}b\left(1 - c_{f\beta} / c_\beta\right)\right] \frac{j_1\left(2k_{f\beta}b\right)}{2k_{f\beta}b}$$

phase correction
The time domain pulse-echo P-wave response of a 1 mm radius spherical inclusion in steel where the density and compressional wave speed are both fifty percent higher than the host steel. Solid line: MBA approximation, dashed line: separation of variables solution.
The time domain pulse-echo P-wave response of a 1 mm radius spherical inclusion in steel where the density and compressional wave speed are both one hundred percent higher than the host steel. Solid line: MBA approximation, dashed line: separation of variables solution.
Flaw Scattering

The Method of Separation of Variables

The sphere and the cylinder are the only two geometries where we can obtain exact separation of variables solutions for elastic wave scattering problems. These are commonly used as "exact" solutions to test more approximate theories and numerical methods.
Flaw Scattering-Void

Example 1: pulse-echo P-wave scattering of a spherical void

\[ A\left( e_i^p ; -e_i^p \right) = \frac{-1}{ik_p} \sum_{n=0}^{\infty} (-1)^n A_n \]

\[ A_n = \frac{E_3E_{42} - E_4E_{32}}{E_{31}E_{42} - E_{41}E_{32}} \]

\[ E_3 = (2n+1) \left\{ \left[ n^2 - n - \left( k_s^2 b^2 / 2 \right) \right] j_n(k_{pb}) + 2k_p b j_{n+1}(k_{pb}) \right\} \]

\[ E_4 = (2n+1) \left\{ (n-1) j_n(k_{pb}) - k_p b j_{n+1}(k_{pb}) \right\} \]

\[ E_{31} = \left[ n^2 - n - \left( k_s^2 b^2 / 2 \right) \right] h_n^{(1)}(k_{pb}) + 2k_p b h_{n+1}^{(1)}(k_{pb}) \]

\[ E_{41} = (n-1) h_n^{(1)}(k_{pb}) - k_p b h_{n+1}^{(1)}(k_{pb}) \]

\[ E_{32} = -n(n+1) \left[ (n-1) h_n^{(1)}(k_s b) - k_s b h_{n+1}^{(1)}(k_s b) \right] \]

\[ E_{42} = -\left[ n^2 - 1 - \left( k_s^2 b^2 / 2 \right) \right] h_n^{(1)}(k_s b) - k_s b h_{n+1}^{(1)}(k_s b) \]
The normalized P-wave pulse-echo scattering amplitude $2A/b$ for a spherical void of radius $b$. 
Using this separation of variables solutions at many frequencies to synthesize a P-wave impulse time domain solution (pulse-echo).

The time-domain pulse-echo P-wave response of a 0.5 mm radius spherical void in steel (c_p = 5900 m/s, c_s = 3200 m/sec) obtained by applying a low-pass cosine-squared windowing filter between 10 and 20 MHz to the separation of variables solution and then inverting the result into the time domain with the inverse Fourier transform.
Flaw Scattering-Void

Example 2: pulse-echo SV-wave scattering of a spherical void

\[ A(e_i^s; -e_i^s) = \frac{-1}{ik_s} \sum_{n=1}^{\infty} \frac{(-1)^n (2n+1) B_n}{2} \]

\[ B_n = \frac{H_{13} J_{42} - H_{43} J_{12}}{H_{13} H_{42} - H_{43} H_{12}} - \frac{J_{41}}{H_{41}} \]

\[ J_{12} = n(n+1) \left[ (n-1) j_n(k_s b) - k_s b j_{n+1}(k_s b) \right] \]

\[ H_{12} = n(n+1) \left[ (n-1) h_n^{(1)}(k_s b) - k_s b h_{n+1}^{(1)}(k_s b) \right] \]

\[ H_{13} = \left[ n^2 - n - \left( k_s^2 b^2 / 2 \right) \right] h_n^{(1)}(k_p b) + 2k_p b h_{n+1}^{(1)}(k_p b) \]

\[ J_{41} = (n-1) j_n(k_s b) - k_s b j_{n+1}(k_s b) \]

\[ H_{41} = (n-1) h_n^{(1)}(k_s b) - k_s b h_{n+1}^{(1)}(k_s b) \]

\[ J_{42} = \left[ n^2 - 1 - \left( k_s^2 b^2 / 2 \right) \right] j_n(k_s b) + k_s b j_{n+1}(k_s b) \]

\[ H_{42} = \left[ n^2 - 1 - \left( k_s^2 b^2 / 2 \right) \right] h_n^{(1)}(k_s b) + k_s b h_{n+1}^{(1)}(k_s b) \]

\[ H_{43} = (n-1) h_n^{(1)}(k_p b) - k_p b h_{n+1}^{(1)}(k_p b) \]
The SV-wave pulse-echo scattering amplitude

The magnitude of the pulse-echo SV-wave response, $A$, versus frequency for a 0.5 mm radius spherical void in steel ($c_p = 5900\, \text{m/s}$, $c_s = 3200\, \text{m/sec}$) as calculated by the method of separation of variables.
Using this separation of variables solutions at many frequencies to synthesize an SV-wave impulse time domain solution (pulse-echo)

The time domain pulse-echo SV-wave response for the same void considered in the P-wave case by applying a low-pass cosine-squared windowing filter between 10 and 20 MHz to the separation of variables solution and then inverting the result into the time domain with the inverse Fourier transform.
Flaw Scattering- SDH

Example 3: pulse-echo P-wave scattering of a cylindrical void

\[ \frac{A_{3D}\left(e_i^p; -e_i^p\right)}{L} = \frac{i}{2\pi} \sum_{n=0}^{\infty} (2 - \delta_{0n})(-1)^n F_n \]

\[ A_{2D}(\omega) = \left(\frac{2i\pi}{k_{\alpha_2}}\right)^{1/2} \frac{A_{3D}(\omega)}{L} \]

\[ \delta_{0n} = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \]

\[ F_n = 1 + \frac{C_n^{(2)}(k_p b) C_n^{(1)}(k_s b) - D_n^{(2)}(k_p b) D_n^{(1)}(k_s b)}{C_n^{(1)}(k_p b) C_n^{(1)}(k_s b) - D_n^{(1)}(k_p b) D_n^{(1)}(k_s b)} \]

\[ C_n^{(i)}(x) = \left(n^2 + n - (k_s b)^2/2\right) H_n^{(i)}(x) - \left(2n H_n^{(i)}(x) - x H_n^{(i)}(x)\right) \]

\[ D_n^{(i)}(x) = n(n+1) H_n^{(i)}(x) - n \left(2n H_n^{(i)}(x) - x H_n^{(i)}(x)\right) \]
Recall, the scattering amplitude for the pulse-echo P-wave case was:

\[ \frac{|A_{3D}|}{L} \]

vs.

\[ k_p b \]
Using this separation of variables solutions at many frequencies to synthesize a P-wave impulse time domain solution (pulse-echo)

The time-domain pulse-echo P-wave response of a 0.5 mm radius cylindrical void in steel (\(c_p = 5900\) m/s, \(c_s = 3200\) m/sec) obtained by applying a low-pass cosine-squared windowing filter between 10 and 20 MHz to the separation of variables solution and then inverting the result into the time domain with the inverse Fourier transform.
Example 4: pulse-echo SV-wave scattering of a cylindrical void

\[
\frac{A_{3D}(e_i^{sv}; -e_i^{sv})}{L} = \frac{i}{2\pi} \sum_{n=0}^{\infty} (2 - \delta_{0n})(-1)^n G_n
\]

\[
\delta_{0n} = \begin{cases} 
1 & n = 0 \\
0 & \text{otherwise}
\end{cases}
\]

\[
G_n = 1 + \frac{C_n^{(2)}(k_s b) C_n^{(1)}(k_p b) - D_n^{(2)}(k_s b) D_n^{(1)}(k_p b)}{C_n^{(1)}(k_p b) C_n^{(1)}(k_s b) - D_n^{(1)}(k_p b) D_n^{(1)}(k_s b)}
\]
Again, the scattering amplitude for the pulse-echo SV-wave case was:
Flaw Scattering - SDH

Using this separation of variables solutions at many frequencies to synthesize an SV-wave impulse time domain solution (pulse-echo)

The time-domain pulse-echo SV-wave response of a 0.5 mm radius cylindrical void in steel ($c_p = 5900$ m/s, $c_s = 3200$ m/sec) obtained by applying a low-pass cosine-squared windowing filter between 10 and 20 MHz to the separation of variables solution and then inverting the result into the time domain with the inverse Fourier transform.
Flaw Scattering - SDH

Experimentally determined scattering amplitude by deconvolution (side-drilled hole)

\[ V_R(\omega) = G(\omega) \left[ \frac{A(\omega)}{L} \right] \]

\[ G(\omega) = s(\omega) E(\omega) \]

\[ E(\omega) = \left[ \int_{L} \hat{V}_0^{(1)}(z, \omega) \hat{V}_0^{(2)}(z, \omega) \, dz \right] \left[ \frac{4\pi \rho c_{\alpha_2}}{-ik_{\alpha_2} Z_{T; a}} \right] \]
Flaw Scattering -SDH