A STUDY OF EXPERIMENTAL DESIGN OPTIMALITY FOR WIENER SYSTEMS
A Study of Experimental Design Optimality for Wiener Systems

AULIA HARDJASAMUDRA,1 DERRICK K. ROLLINS,2 NIDHI BHANDARI,2 AND SWEE-TENG CHIN2

1Department of Statistics, Iowa State University, Ames Iowa
2Department of Chemical and Biological Engineering, Iowa State University, Ames Iowa

In the context of nonlinear dynamic system identification for Hammerstein systems, Rollins et al. (2003a) studied the information efficiency of the following two competing experimental design approaches: statistical design of experiments (SDOE) and pseudo-random sequences design (PRSD). The focus of this study is the Wiener system and evaluates SDOE against PRS under D-optimal efficiency. Three cases are evaluated and the results strongly support SDOE as the better approach.

Keywords Block-oriented modeling; D-optimality; Dynamic modeling; Nonlinear dynamic systems; Semi-empirical modeling; Statistical design of experiment; Wiener systems

Introduction

In industry, effective process control is important in meeting product quality specifications while achieving production rate requirements. In order to control a process, it is helpful to understand the process behavior, and model-based approaches are quite popular. The basic goal of all model-based approaches is to find accurate causal relationships between process inputs and the outputs. Commonly, model development relies on performing experiments on the real process; thus, optimal experimental design is essential.

Since the experimental design used to collect data has a big impact on model accuracy, the design should be chosen carefully to make sure that high information content is achieved in the resulting data. Statistical design of experiments (SDOE) has the ability to provide input sequences that minimize the uncertainty in the parameter estimates. SDOE is widely used in many data collection fields, whereas pseudo-random sequence design (PRSD) is used almost exclusively in engineering system identification (Henson and Seborg, 1997). To illustrate the difference between input sequences based on SDOE and PRSD, an example of each design is shown in Figure 1. The SDOE has a smaller number of input changes as a series of sequential step tests (i.e., the process reaches steady state for each set of input changes or design points). The PRSD has a much larger number of input changes, with very few (if any) steady-state conditions.

Address correspondence to Derrick K. Rollins, Department of Chemical and Biological Engineering, Iowa State University, Ames, IA 50011. E-mail: drollins@iastate.edu
Therefore, the main objective of this study is to compare SDOE and PRSD information content in estimating parameters for nonlinear dynamic block-oriented systems with a Wiener structure. A quantitative measure of efficiency based on D-optimality will be used to determine the information content of each experimental design; its definition is presented in the next section. The following section presents the closed-form exact solution for the Wiener system applicable to this work. Three case studies are then presented with two, five, and seven inputs, respectively. Finally, we give a summary and make closing remarks.

Efficiency

Efficiency based on D-optimality is used in this work to quantitatively compare the information content of two competing experimental designs in the Wiener process. The D-optimal criterion minimizes the generalized variance of the parameters in the process. In other words, this criterion minimizes the width of the confidence interval for the parameter estimates (Myers and Montgomery, 1995) and is used to provide approximate inference regions (see Bates and Watts, 1988). D-optimal efficiency for two competing designs, D1 and D2, is a ratio of determinants as shown by:

\[
\eta = \frac{\left| V^{T}V \right|_{D2}^{1/p}}{\left| V^{T}V \right|_{D1}^{1/p}} \times 100
\]

where \( p \) is the number of model parameters, \( N_{D1} \) and \( N_{D2} \) are the numbers of experimental points in Design 1 and Design 2, respectively, and \( V \) is the derivative matrix of the nonlinear model with \((N \times p)\) elements, defined below:

\[
V = \begin{bmatrix}
\frac{\partial \xi}{\partial \theta_1} |_{1} & \cdots & \frac{\partial \xi}{\partial \theta_1} |_{1} & \cdots & \frac{\partial \xi}{\partial \theta_p} |_{1} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\frac{\partial \xi}{\partial \theta_N} |_{1} & \cdots & \frac{\partial \xi}{\partial \theta_N} |_{1} & \cdots & \frac{\partial \xi}{\partial \theta_N} |_{N}
\end{bmatrix}
\]
samples (Bates and Watts, 1988). The elements in $V$ measure the sensitivity of the process model parameters, $\theta_i$. The D-optimal criterion affects the quality of the parameter estimates by maximizing the determinant of $V^TV$. Maximizing $|V^TV|$ is equivalent to minimizing the volume of the confidence ellipsoid or the width of the confidence interval of the parameters.

D-optimal efficiency described in Equation (2) gives a quantitative measurement of the information content of the competing designs. If the ratio is less than 100%, Design 2 is more efficient than Design 1. To calculate the derivative matrix for each design, the analytical derivatives are needed in Equation (2). In this work the true output response, $\xi(t)$, is modeled using a block-oriented Wiener system and applying the method in Bhandari and Rollins (2003). In the next section, details of this model will be discussed.

The Wiener System

The block structure of a multiple-input, single-output (MISO) Wiener system is shown in Figure 2. There are different dynamic blocks associated with each input. Each of the q inputs, $u_k$, passes through a separate linear dynamic block, $G_k$, and produces an intermediate variable, $v_k$, that is an element of the vector $v$. Then $v$ passes through a nonlinear static function, $f(v)$, and generates the output, $\xi(t)$. Two advantages of the Wiener system that contribute to its popularity are:

1. It allows each input to have separate linear dynamics, and
2. It is able to treat nonlinear dynamic systems through nonlinear functions of the linear dynamic outputs from the first blocks.

Rollins et al. (1998, 2003b) proposed a closed-form exact solution to the Hammerstein system that they call H-BEST (Hammerstein block-oriented exact solution technique). Bhandari and Rollins (2003) extended this approach to multiple-input, multiple-output (MIMO) Wiener systems and called it the Wiener block-oriented exact solution technique or W-BEST. Because it is given in closed form, the W-BEST

![Figure 2. A general “MISO Wiener model” structure with $k = 1, 2, \ldots, q$ inputs.](image)
algorithm will be used throughout this article to calculate the elements in the derivative matrix, \( \mathbf{V} \).

The input sequence applicable to W-BEST consists of step changes occurring at times \( t = 0, t_1, t_2, \ldots \), as shown in Equation (3) where \( u_{k,i} \) is the value of the \( k \)th input in the \( i \)th interval.

\[
\begin{align*}
t = 0, & \quad u_k(t) = 0 \\
0 < t \leq t_1, & \quad u_k(t) = u_{k,1} \\
\vdots & \quad \vdots \\
t_{i-1} < t \leq t_i, & \quad u_k(t) = u_{k,i} \\
\vdots & \quad \vdots
\end{align*}
\]  
\( (3) \)

The W-BEST algorithm for the input changes in Equation (3) is given by Equations (4) and (5):

\[
\begin{align*}
t = 0, & \quad v_k(t) = 0 \\
0 < t \leq t_1, & \quad v_k(t) = u_{k,1} \cdot g_k(t; \tau_k) \\
\vdots & \quad \vdots \\
t_{i-1} < t \leq t_i, & \quad v_k(t) = v_k(t_{i-1}) + \left[ (u_{k,i} - v_k(t_{i-1})) \cdot g_k(t - t_{i-1}; \tau_k) \right] \\
\vdots & \quad \vdots
\end{align*}
\]  
\( (4) \)

where \( v_i \) is the intermediate variables for the \( k \)th input \( g_k(t; \tau_k) = \mathcal{L}^{-1}[G_k(s; \tau_k)/s] \) and \( G_k(s; \tau_k) = v_k(s)/u_k(s) \). The dynamic parameter of the \( k \)th input is \( \tau_k \) and \( \mathcal{L}^{-1} \) is the inverse Laplace transform operator. The true response variable is

\[
\xi(t) = f(v(t))
\]  
\( (5) \)

where \( v = [v_1, v_2, \ldots, v_k, \ldots, v_q]^T \) and \( f(\cdot) \) is the static gain function. The solution given above is an exact solution under certain restrictions. Details of these conditions and a mathematical proof are presented in Bhandari and Rollins (2003). However, when the linear dynamic function is first order, the only restriction is that the input changes be step changes, which is the case for this work.

**Case Studies**

The efficiency of SDOE and PRSD are compared based on three different case studies. The first case study consists of two input variables, the second consists of five input variables, and the third consists of seven input variables. In each case, SDOE and PRSD will be used to generate a series of sequential step input changes. The true response for the system is generated using a mathematical Wiener system. The partial derivatives terms for the derivative matrix, \( \mathbf{V} \), for each experimental design will be calculated using the closed-form W-BEST algorithm given in Equations (4) and (5). Then, Equation (1) will be used to determine the efficiencies of competing experimental designs.

**Two Input Variables**

The Wiener system in the first case study consists of two input variables, \( u_1(t) \) and \( u_2(t) \), and a single output. A first-order linear dynamic block represents the dynamics
for each input, and the static gain function is given by a second-order polynomial with quadratic and interaction terms. The first-order dynamic functions are given by Equations (6) and (7):

\[ G_k(s) = \frac{1}{\tau_k s + 1} \]  

\[ g_k(t; \tau_k) = \mathcal{L}^{-1} \left[ \frac{1}{\tau_k s + 1} \cdot \frac{1}{s} \right] = 1 - e^{-\frac{t}{\tau_k}} \]  

where \( k = 1, 2, \tau_1 = 5, \) and \( \tau_2 = 3. \) The second-order static gain (steady state) function is given by:

\[ \xi(t) = f\{v(t)\} = a_1 v_1(t) + a_2 v_2(t) + a_3 [v_1(t)]^2 + a_4 [v_2(t)]^2 + a_5 v_1(t) v_2(t) \]  

where parameters \( a_1 \) to \( a_5 \) are 2.25, 1.75, 0.25, 0.50, and 1.5, respectively. The derivative matrix, \( V, \) for this system is:

\[
V = \begin{bmatrix}
\frac{\partial \xi}{\partial a_1} & \cdots & \frac{\partial \xi}{\partial a_5} & \frac{\partial \xi}{\partial \tau_1} & \frac{\partial \xi}{\partial \tau_2}
\end{bmatrix}
\]

where the elements of \( V \) are obtained by partial differentiation of the exact W-BEST solution above and are shown in Equations (10)–(27). For \( 0 < t < t_1, \)

\[ \frac{\partial \xi(t)}{\partial a_1} = v_1(t) = u_{1,1} \left( 1 - e^{\frac{-t}{\tau_1}} \right) \]  

\[ \frac{\partial \xi(t)}{\partial a_2} = v_2(t) = u_{2,1} \left( 1 - e^{\frac{-t}{\tau_2}} \right) \]  

\[ \frac{\partial \xi(t)}{\partial a_3} = [v_1(t)]^2 = \left[ u_{1,1} \left( 1 - e^{\frac{-t}{\tau_1}} \right) \right]^2 \]  

\[ \frac{\partial \xi(t)}{\partial a_4} = [v_2(t)]^2 = \left[ u_{2,1} \left( 1 - e^{\frac{-t}{\tau_2}} \right) \right]^2 \]  

\[ \frac{\partial \xi(t)}{\partial a_5} = [v_1(t)] \cdot [v_2(t)] \]  

\[ = \left[ u_{1,1} \left( 1 - e^{\frac{-t}{\tau_1}} \right) \right] \cdot \left[ u_{2,1} \left( 1 - e^{\frac{-t}{\tau_2}} \right) \right] \]  

\[ \frac{\partial \xi(t)}{\partial \tau_1} = [a_1 + 2a_3 v_1(t) + a_5 v_2(t)] \frac{\partial v_1(t)}{\partial \tau_1} \]  

\[ \frac{\partial \xi(t)}{\partial \tau_2} = [a_2 + 2a_4 v_2(t) + a_5 v_1(t)] \frac{\partial v_2(t)}{\partial \tau_2} \]
where

\[
\frac{\partial v_1(t)}{\partial \tau_1} = u_{1,1} \left( -e^{\tau_2} \right) \left( \frac{t - t_1}{\tau_1} \right) \tag{17}
\]

\[
\frac{\partial v_2(t)}{\partial \tau_2} = u_{2,1} \left( -e^{\tau_2} \right) \left( \frac{t - t_1}{\tau_2} \right) \tag{18}
\]

and for \( t_1 < t \leq t_2 \),

\[
\frac{\partial \xi(t)}{\partial a_1} = v_1(t) = v_1(t_1) + (u_{1,2} - v_1(t_1)) \left( 1 - e^{-\frac{(t-t_1)}{\tau_1}} \right) \tag{19}
\]

\[
\frac{\partial \xi(t)}{\partial a_2} = v_2(t) = v_2(t_1) + (u_{2,2} - v_2(t_1)) \left( 1 - e^{-\frac{(t-t_2)}{\tau_2}} \right) \tag{20}
\]

\[
\frac{\partial \xi(t)}{\partial a_3} = [v_1(t)]^2 = \left[ v_1(t_1) + (u_{1,2} - v_1(t_1)) \left( 1 - e^{-\frac{(t-t_1)}{\tau_1}} \right) \right]^2 \tag{21}
\]

\[
\frac{\partial \xi(t)}{\partial a_4} = [v_2(t)]^2 = \left[ v_2(t_1) + (u_{2,2} - v_2(t_1)) \left( 1 - e^{-\frac{(t-t_2)}{\tau_2}} \right) \right]^2 \tag{22}
\]

\[
\frac{\partial \xi(t)}{\partial a_5} = [v_1(t)] \cdot [v_2(t)]
\]

\[
= \left[ v_1(t_1) + (u_{1,2} - v_1(t_1)) \left( 1 - e^{-\frac{(t-t_1)}{\tau_1}} \right) \right] x
\]

\[
\left[ v_2(t_1) + (u_{2,2} - v_2(t_1)) \left( 1 - e^{-\frac{(t-t_2)}{\tau_2}} \right) \right]
\]

\[
\frac{\partial \xi(t)}{\partial \tau_1} = [a_1 + 2a_3 v_1(t) + a_5 v_2(t)] \frac{\partial v_1(t)}{\partial \tau_1} \tag{24}
\]

\[
\frac{\partial \xi(t)}{\partial \tau_2} = [a_2 + 2a_4 v_2(t) + a_5 v_1(t)] \frac{\partial v_2(t)}{\partial \tau_2} \tag{25}
\]

where

\[
\frac{\partial v_1(t)}{\partial \tau_1} = \left. \left[ \frac{\partial v_1(t)}{\partial \tau_1} \right] \right|_{t_1} - (u_{1,2} - v_1(t_1)) \left( \frac{t - t_1}{\tau_1} \right) e^{-\frac{(t-t_1)}{\tau_1}} \tag{26}
\]

\[
\frac{\partial v_2(t)}{\partial \tau_2} = \left. \left[ \frac{\partial v_2(t)}{\partial \tau_2} \right] \right|_{t_1} - (u_{2,2} - v_2(t_1)) \left( \frac{t - t_1}{\tau_2} \right) e^{-\frac{(t-t_1)}{\tau_2}} \tag{27}
\]

and \( \frac{\partial v_1(t)}{\partial \tau_1} \bigg|_{t_1} \) is the partial derivative of the intermediate variable \( v_1(t) \) with respect to \( \tau_1 \) evaluated at time \( t_1 \), and \( \frac{\partial v_2(t)}{\partial \tau_2} \bigg|_{t_1} \) is the partial derivative of the intermediate variable \( v_2(t) \) with respect to \( \tau_2 \) evaluated at time \( t_1 \). The derivatives of the subsequent intervals are calculated in a similar fashion.
A full factorial design (see Montgomery, 1984) is chosen for the input sequence generation for SDOE. Since the input has three levels, $-1$, $0$, and $1$, a three-level factorial design is used. The $3^2$ design matrix is given by

$$
\mathbf{B}^T = [\mathbf{u}_1 \mathbf{u}_2]^T = 
\begin{bmatrix}
-1 & -1 & -1 & 0 & 0 & 1 & 1 & 1 \\
-1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1
\end{bmatrix}
$$

(28)

The amount of time needed by the system to approximately reach steady state for each step change in the SDOE determines the total time length of the design. The total experimental time is kept identical for both designs. Typically, the time needed for a first-order process to reach 99.3% of steady state is $5\tau$, while a time of $4\tau$ allows it to reach 98.2% of the steady-state response. Rollins et al. (2003a, 2006) found that the input change time of each design point could be reduced to $4\tau$ without losing significant information for model accuracy. Therefore, the run time we used for each input step change in SDOE was $4\tau$. Since $\tau_1 = 5$, the time between input step changes is $4 \times 5$ or 20 time units. Therefore, because this chosen SDOE has nine design points or experimental runs, the total time for this SDOE is 180 time units. The SDOE input sequence based on the design matrix given in Equation (28) is shown in Figure 3.

The following modified Ljung (1999) equation is used to generate the input sequences for PRSD in this case:

$$
U(t) = \text{rem}(A(q)U(t), 3) - 1 = \text{rem}(a_1U(t-1) + \cdots + a_nU(t-n), 3)
$$

(29)

where $U(0) = 1$, and $U(t)$, for $t > 1$, is the input level with normalized values of 0, 1, or 2, $\text{rem}(x, 3)$ is the remainder as $x$ is divided by 3, $n$ is the order of the equation and determines the maximum period length, and the input, $u(t) = U(t) - 1$. The coefficients $a_1, \ldots, a_n$, which depend on the order $n$, are available in Ljung (1999). For example, when $n = 9$, $a_i = 1$ for $i = 4$, and 9, and zero otherwise. Since PRSD input generation is pseudo-random, a large number of different input sequences for PRSD can be obtained. Different input sequences for PRSD can result in different responses. Therefore, six different input sequences for PRSD are used in

![Figure 3. Input sequence based on SDOE for a two-input system.](image-url)
this study, all with a switching time of five time units (which equals the time constant for this process), to assess the variability of the results. Each PRSD is compared to SDOE to determine its efficiency. For space consideration, only one of the input sequences from PRSD is shown in Figure 4.

The data are sampled every 0.5 time units, generating a V with 360 rows, which is equal to the number of total samples. The efficiency results are presented in Table I. The results show that SDOE is a more efficient design than PRSD in terms of providing more information in accurately estimating the model parameters. The PRSD efficiency ranges from 13% to 18% with an average of about 15%. As mentioned before, SDOE satisfies the orthogonality (non-confounding of inputs) requirement and also allows each step test to run long enough for the process to reach steady state, while PRSD does not have these characteristics, as discussed previously. Since both attributes play a critical role in maximizing information content, the results in Table I are consistent with this reasoning.

**Five Input Variables**

The Wiener system in the second case study consists of five input variables, \( u_1(t) \), \( u_2(t) \), \( u_3(t) \), \( u_4(t) \), and \( u_5(t) \), and a single output. The same dynamic structure and

![Figure 4. Input sequence generated by using the modified Ljung equation for a two-input system.](image)

**Table I.** Comparison of design efficiency results for the two-input case study

<table>
<thead>
<tr>
<th>Design</th>
<th>% Efficiency = PRSD/SDOE</th>
</tr>
</thead>
<tbody>
<tr>
<td>SDOE Full factorial</td>
<td>100</td>
</tr>
<tr>
<td>1</td>
<td>16.73</td>
</tr>
<tr>
<td>2</td>
<td>14.75</td>
</tr>
<tr>
<td>3</td>
<td>17.48</td>
</tr>
<tr>
<td>PRSD Modified Ljung</td>
<td>13.67</td>
</tr>
<tr>
<td>4</td>
<td>15.20</td>
</tr>
<tr>
<td>5</td>
<td>13.45</td>
</tr>
<tr>
<td>Average</td>
<td>15.26</td>
</tr>
</tbody>
</table>
static gain functions are used, and the values of the parameters are $\tau_1 = 5$, $\tau_2 = 4.5$, $\tau_3 = 4$, $\tau_4 = 3.5$, $\tau_5 = 3$ and $a_1$ through $a_5$ are 2.4, 1.2, 2.6, 0.5, 1.5, 0.7, 3.2, 1.1, 4.8, 0.6, $-3.0, 5.0, 6.7, -4.5, 2.3, 7.8, 5.4, 4.9, -2.3,$ and $6.0$, respectively.

A Box-Behnken design (Box and Behnken, 1960) is used for the input sequence generation for SDOE in this case. The input sequence has three levels, $-1$, 0, and 1, and a total of five inputs. Therefore, the total number of experimental runs for five input variables is $40 + n_c$, where $n_c$ is the number of center points, which is six for this case. A total of 46 experimental runs in this Box-Behnken design are used to generate the input sequence of SDOE in this study (see Myers and Montgomery, 1995). The run time for each SDOE step test (i.e., design point) was 20 time units, which is based on the largest time constant. The total experimental time for SDOE and PRS designs was 920 time units in this case.

As in the previous study, the modified Ljung equation (i.e., Equation (29)) is used to generate the input sequences for PRSD. Six different input sequences of the PRSD are also used in this case. The data are sampled every 0.5 time units, generating a $V$ with 25 columns and 1840 rows. The efficiency results for five input variables are presented in Table II.

The results show that SDOE is considerably more efficient than PRSD in terms of information content. The efficiency values range from 21% to 39%, with an average of about 30% based on the six different PRSDs. We believe the lack of input orthogonality and steady-state conditions for PRSD contributed greatly to the low efficiencies as in the previous case.

Seven Input Variables

The same Wiener system is used in the third case study with the exception of seven input variables, $u_1(t)$, $u_2(t)$, $u_3(t)$, $u_4(t)$, $u_5(t)$, $u_6(t)$, and $u_7(t)$. The values of the parameters are $\tau_1$ through $\tau_7$, 5.0, 4.8, 4.2, 4.0, 3.7, 3.5, 3.0, respectively, and $a_1$ through $a_{35}$, 2.4, 1.2, 2.6, 0.5, 1.5, 0.7, 3.2, 1.1, 4.8, 0.6, $-3.0, 5.0, 6.7, -4.5, 2.3, 7.8, 5.4, 4.9, -2.3, 6.0, 0.5, 1.2, -0.7, 2.0, 1.0, 1.8, 1.3, 0.9, 3.4, 1.8, 2.9, -3.1, 0.4, 1.6, 2.7, and 5.0, respectively. For example $a_1 = 2.4, a_2 = 1.2, \ldots, a_{35} = 2.7$.

A Box-Behnken design is again used for the input sequence generation for SDOE in this case. The number of Box-Behnken design points for seven inputs (factors) are $56 + n_c$, with $n_c = 6$ in this case. A number of experimental runs of 62 is used to generate the input sequences of SDOE (see Myers and Montgomery, 1995).
The run time for each input step in SDOE is $4\tau$ based on the largest time constant, which is equal to five in this case, and the total experimental time for SDOE and PRS designs was 1240 time units.

The input levels for PRSD is randomly selected from $-1$, 0, or 1 in this case study. A switching probability of 40% is chosen, which means that at every other sampling time there is a 40% probability that the input is switched to a new value. This condition is used to generate the input sequences for six different PRSDs in this case. Again, the data are sampled every 0.5 time units, generating a $V$ with 42 columns and 2480 rows. The efficiency results for seven input variables are presented in Table III.

As in the previous studies, the results show that SDOE is considerably more efficient than PRSD in terms of information content. The efficiency range for the PRSD as compared to SDOE lies between 18% and 27% with an average of 23%. Again, we believe that these results are due to the PRSD limitation of not providing enough steady-state information.

**Concluding Remarks**

In this work, a quantitative measure of information content using D-optimal efficiency was used to compare two competing designs, statistical design of experiments (SDOE) and pseudo-random sequence design (PRSD). This study investigated three different number inputs on a Wiener system. For all the cases, the total time length of the sequences was kept the same for both designs.

The same conclusion was supported in all three case studies: SDOE was considerably more efficient than PRSD. More specifically, these results confirm that SDOE can provide a higher level of information content than PRSD. We are not surprised by this conclusion since the input sequence in PRSD is not orthogonal with respect to design points. The other PRSD limitation supporting this conclusion is the lack of sufficient steady-state data for PRSD design points, which limits the amount of information for estimating the ultimate response parameters in the nonlinear static functions. As mentioned in previously, for a similar Hammerstein study, Rollins et al., (2006) found similar results in support of SDOE over PRSD.

This work can be extended to other types of block-oriented models, such as Sandwich models and Reverse Sandwich models, and also to higher order dynamic functions, such as second-order, second-order plus lead, and so on. In a MIMO

### Table III. Comparison of design efficiency results for the seven-input case study

<table>
<thead>
<tr>
<th>Design</th>
<th>% Efficiency = PRSD/SDOE</th>
</tr>
</thead>
<tbody>
<tr>
<td>SDOE Box-Behnken</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>3</td>
</tr>
<tr>
<td>PRSD Random switching</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>Average</td>
</tr>
</tbody>
</table>

The run time for each input step in SDOE is $4\tau$ based on the largest time constant, which is equal to five in this case, and the total experimental time for SDOE and PRS designs was 1240 time units.

The input levels for PRSD is randomly selected from $-1$, 0, or 1 in this case study. A switching probability of 40% is chosen, which means that at every other sampling time there is a 40% probability that the input is switched to a new value. This condition is used to generate the input sequences for six different PRSDs in this case. Again, the data are sampled every 0.5 time units, generating a $V$ with 42 columns and 2480 rows. The efficiency results for seven input variables are presented in Table III.

As in the previous studies, the results show that SDOE is considerably more efficient than PRSD in terms of information content. The efficiency range for the PRSD as compared to SDOE lies between 18% and 27% with an average of 23%. Again, we believe that these results are due to the PRSD limitation of not providing enough steady-state information.
simulated CSTR study with nonlinear static and second-order plus lead dynamic behavior, Bhandari and Rollins (2003) obtained considerably better fit using by SDOE over PRSD. While these results cannot be generalized to a conclusion that SDOE is superior in all cases, they do supply enough evidence to indicate that SDOE merits consideration as the superior approach as we encourage modelers to examine their particular situations under D-optimal efficiency. Moreover, other criteria, such as A-optimal or G-optimal, can be used in a similar fashion to compare other characteristics of the competing designs.

References