STAT611 **Homework 7** **Solutions**

1. (a)

$$\underline{y} = \underline{1}\beta + \underline{x}b + \underline{e} \qquad \mathrm{Var}(\underline{y}) = \sigma_b^2 \underline{x}\underline{x}' + \sigma^2 I$$

(b) i. By Problem 3 on Exam 2, the quadratic form will be χ^2 distributed if "$A\Sigma$" is idempotent. Note that

$$\frac{1}{\sigma^2}(P_X - P_{\underline{x}})(\sigma_b^2 \underline{x}\underline{x}' + \sigma^2 I) = (P_X - P_{\underline{x}})(\frac{\sigma_b^2}{\sigma^2}\underline{x}\underline{x}' + I) = P_X - P_{\underline{x}},$$

which is idempotent. Thus, the quadratic form is χ^2 distributed. Now

$$\mathrm{rank}[(P_X - P_{\underline{x}})/\sigma^2] = 1, \text{ and}$$

$$\frac{1}{\sigma^2}\beta^2 \underline{1}'(P_X - P_{\underline{x}})\underline{1} = \frac{n\beta^2 \sum_{i=1}^n (x_i - \bar{x})^2}{\sigma^2 \sum_{i=1}^n x_i^2}.$$

Thus, the quadratic form is noncentral χ_1^2 with noncentrality parameter

$$\frac{n\beta^2 \sum_{i=1}^n (x_i - \bar{x})^2}{\sigma^2 \sum_{i=1}^n x_i^2}.$$

ii. By Problem 3 on Exam 2, the quadratic form will be χ^2 distributed if "$A\Sigma$" is idempotent. Note that

$$\frac{1}{\sigma^2}(I - P_X)(\sigma_b^2 \underline{x}\underline{x}' + \sigma^2 I) = (I - P_X)(\frac{\sigma_b^2}{\sigma^2}\underline{x}\underline{x}' + I) = I - P_X,$$

which is idempotent. Thus, the quadratic form is χ^2 distributed. Now

$$\mathrm{rank}[(I - P_X)/\sigma^2] = n - 2, \text{ and}$$

$$\frac{1}{\sigma^2}\beta^2 \underline{1}'(I - P_X)\underline{1} = 0.$$

Thus, the quadratic form is central χ_{n-2}^2.

iii.

$$(P_X - P_{\underline{x}})(\sigma_b^2 \underline{x}\underline{x}' + \sigma^2 I)(I - P_X)/\sigma^4 = (P_X - P_{\underline{x}})(I - P_X)/\sigma^2 = 0.$$

Thus, by Theorem Q2 from the notes, these quadratic forms are independent.

(c) i. $(n-2)\underline{y}'(P_X - P_{\underline{x}})\underline{y}/\underline{y}'(I - P_X)\underline{y}$.

ii. Central F with 1 and $n-2$ d.f.

iii. Noncentral F with 1 and $n-2$ d.f. and ncp from part (b)i.

iv. Find the probability that a central F random variable with 1 and $n-2$ d.f. would exceed the observed value of $(n-2)\underline{y}'(P_X - P_{\underline{x}})\underline{y}/\underline{y}'(I-P_X)\underline{y}$.

2. (a) $y_1 = \underline{x}_1'\underline{\beta} + \varepsilon_1 = \underline{\ell}'\underline{\beta} + u = \tau$, where $\underline{\ell}' = \underline{x}_1'$ and $\varepsilon_1 = u$. The general formula for the BLUP of τ is $\hat{\tau} = \underline{\ell}'\hat{\underline{\beta}} + \hat{u}$, where $\hat{\underline{\beta}}$ is any solution to the Aitken equations and $\hat{u} = M'W^{-1}(\underline{y} - X\hat{\underline{\beta}})$ where $\mathrm{Cov}(\underline{\varepsilon}, u) = \sigma^2 M$. In this case, X_2 plays the role of X, $\underline{y}_2$ plays the role of $\underline{y}$, $W = I$, and $M = \underline{0}$. Thus $\hat{\tau}$ is $\underline{x}_1'(X_2'X_2)^{-1}X_2'\underline{y}_2$.

(b) Note that $X'X = X_2'X_2 + \underline{x}_1\underline{x}_1'$. Thus, Problem 9 from HW 3,

$$(X'X)^{-1} = (X_2'X_2)^{-1} - \frac{(X_2'X_2)^{-1}\underline{x}_1\underline{x}_1'(X_2'X_2)^{-1}}{1 + \underline{x}_1'(X_2'X_2)^{-1}\underline{x}_1}.$$

It follows that

$$\frac{1}{1 - \underline{x}_1'(X'X)^{-1}\underline{x}_1} = 1 + \underline{x}_1'(X_2'X_2)^{-1}\underline{x}_1$$

and that

$$\underline{x}_1'(X'X)^{-1}X'\underline{y} = \underline{x}_1'(X_2'X_2)^{-1}X'\underline{y} - \frac{\underline{x}_1'(X_2'X_2)^{-1}\underline{x}_1\underline{x}_1'(X_2'X_2)^{-1}X'\underline{y}}{1 + \underline{x}_1'(X_2'X_2)^{-1}\underline{x}_1}.$$

Thus

$$\begin{aligned}
\frac{y_1 - \hat{y}_1}{1 - \underline{x}_1'(X'X)^{-1}\underline{x}_1} &= [1 + \underline{x}_1'(X_2'X_2)^{-1}\underline{x}_1]y_1 - [1 + \underline{x}_1'(X_2'X_2)^{-1}\underline{x}_1]\underline{x}_1'(X_2'X_2)^{-1}X'\underline{y} \\
&\quad + \underline{x}_1'(X_2'X_2)^{-1}\underline{x}_1\underline{x}_1'(X_2'X_2)^{-1}X'\underline{y} \\
&= y_1 + \underline{x}_1'(X_2'X_2)^{-1}\underline{x}_1 y_1 - \underline{x}_1'(X_2'X_2)^{-1}X'\underline{y} \\
&= y_1 + \underline{x}_1'(X_2'X_2)^{-1}\underline{x}_1 y_1 - \sum_{i=1}^{n} \underline{x}_1'(X_2'X_2)^{-1}\underline{x}_i y_i \\
&= y_1 - \sum_{i=2}^{n} \underline{x}_1'(X_2'X_2)^{-1}\underline{x}_i y_i \\
&= y_1 - \underline{x}_1'(X_2'X_2)^{-1}X_2'\underline{y}_2 = d_1.
\end{aligned}$$

(c) $d_1 = [1, -\underline{x}_1'(X_2'X_2)^{-1}X_2']\underline{y}$. Thus d_1 is normally distributed with mean

$$[1, -\underline{x}_1'(X_2'X_2)^{-1}X_2']X\underline{\beta} = \underline{x}_1'\underline{\beta} - \underline{x}_1'(X_2'X_2)^{-1}X_2'X_2\underline{\beta} = 0$$

and variance

$$\begin{aligned}
\sigma^2[1, -\underline{x}_1'(X_2'X_2)^{-1}X_2'][1, -\underline{x}_1'(X_2'X_2)^{-1}X_2']' &= \sigma^2(1 + \underline{x}_1'(X_2'X_2)^{-1}X_2'X_2(X_2'X_2)^{-1}\underline{x}_1) \\
&= \sigma^2(1 + \underline{x}_1'(X_2'X_2)^{-1}\underline{x}_1).
\end{aligned}$$

(d) From the previous part, we have

$$\frac{d_1}{\sqrt{\sigma^2(1+\underline{x}_1'(X_2'X_2)^{-1}\underline{x}_1)}} \sim N(0,1).$$

We know that $(n-p-1)\hat{\sigma}^2_{(1)}/\sigma^2 \sim \chi^2(n-p-1)$ from our course notes. Note that

$$\hat{\sigma}^2_{(1)} = \underline{y}' \begin{bmatrix} 0 & \underline{0}' \\ \underline{0} & I - P_{X_2} \end{bmatrix} \underline{y}/(n-p-1) \text{ and } [1, -\underline{x}_1'(X_2'X_2)^{-1}X_2'] \begin{bmatrix} 0 & \underline{0}' \\ \underline{0} & I - P_{X_2} \end{bmatrix} = \underline{0}'.$$

Thus

$$\frac{d_1}{\sqrt{\sigma^2(1+\underline{x}_1'(X_2'X_2)^{-1}\underline{x}_1)}} \text{ and } (n-p-1)\hat{\sigma}^2_{(1)}/\sigma^2$$

are independent by Theorem A.88. Thus

$$\frac{d_1/\sqrt{\sigma^2(1+\underline{x}_1'(X_2'X_2)^{-1}\underline{x}_1)}}{\sqrt{[(n-p-1)\hat{\sigma}^2_{(1)}/\sigma^2]/(n-p-1)}} = \frac{d_1}{\hat{\sigma}_{(1)}\sqrt{1+\underline{x}_1'(X_2'X_2)^{-1}\underline{x}_1}} \sim t(n-p-1).$$

(e) If $\mathrm{Var}(\varepsilon_1) = \sigma^2 + \sigma^2_\delta$, then

$$\frac{d_1}{\sqrt{\sigma^2_\delta + \sigma^2(1+\underline{x}_1'(X_2'X_2)^{-1}\underline{x}_1)}} \sim N(0,1)$$

independent of $(n-p-1)\hat{\sigma}^2_{(1)}/\sigma^2 \sim \chi^2(n-p-1)$. Thus

$$\begin{aligned}
\frac{d_1}{\hat{\sigma}_{(1)}\sqrt{1+\underline{x}_1'(X_2'X_2)^{-1}\underline{x}_1}} &= \frac{d_1/\sqrt{\sigma^2(1+\underline{x}_1'(X_2'X_2)^{-1}\underline{x}_1)}}{\sqrt{[(n-p-1)\hat{\sigma}^2_{(1)}/\sigma^2]/(n-p-1)}} \\
&= \left(\sqrt{\frac{\sigma^2_\delta + \sigma^2(1+\underline{x}_1'(X_2'X_2)^{-1}\underline{x}_1)}{\sigma^2(1+\underline{x}_1'(X_2'X_2)^{-1}\underline{x}_1)}}\right) \frac{d_1/\sqrt{\sigma^2_\delta + \sigma^2(1+\underline{x}_1'(X_2'X_2)^{-1}\underline{x}_1)}}{\sqrt{[(n-p-1)\hat{\sigma}^2_{(1)}/\sigma^2]/(n-p-1)}} \\
&\overset{d}{=} \left(\sqrt{1 + \frac{\sigma^2_\delta}{\sigma^2(1+\underline{x}_1'(X_2'X_2)^{-1}\underline{x}_1)}}\right) t(n-p-1).
\end{aligned}$$

Thus under H_0, the test statistic from (d) (call it t) will have a central t-distribution with $n-p-1$ degrees of freedom. Under the alternative, t is distributed as a constant greater than one times a central t-distribution with $n-p-1$ degrees of freedom. Thus

$$P_{H_A}(|t| > x) > P_{H_0}(|t| > x)\ \forall\, x > 0.$$

Thus we can compute a p-value by comparison with a $t(n-p-1)$ distribution as in any two-sided t-test.

3. Suppose $y_1, \ldots, y_n \sim N(\mu, \sigma^2)$.

(a) $[y_2 - \bar{y}, \ldots, y_n - \bar{y}]'$.

(b) $[y_2 - \bar{y}, \ldots, y_n - \bar{y}]' = A'\underline{y}$ where

$$A' = [\underline{0}_{n-1}, I_{(n-1)\times(n-1)}] - \frac{1}{n}\underline{1}_{n-1}\underline{1}_n'.$$

Thus $[y_2 - \bar{y}, \ldots, y_n - \bar{y}]'$ is normally distributed with mean

$$\left([\underline{0}_{n-1}, I_{(n-1)\times(n-1)}] - \frac{1}{n}\underline{1}_{n-1}\underline{1}_n'\right)\underline{1}_n\mu = \underline{1}_{n-1}\mu - \underline{1}_{n-1}\mu = \underline{0}$$

and variance

$$\sigma^2\left([\underline{0}_{n-1}, I_{(n-1)\times(n-1)}] - \frac{1}{n}\underline{1}_{n-1}\underline{1}_n'\right)\left([\underline{0}_{n-1}, I_{(n-1)\times(n-1)}] - \frac{1}{n}\underline{1}_{n-1}\underline{1}_n'\right)'$$
$$= \sigma^2\left(I_{(n-1)\times(n-1)} - \frac{1}{n}\underline{1}_{n-1}\underline{1}_{n-1}'\right) \equiv \sigma^2 W.$$

(c)

$$L(\sigma^2; A'\underline{y}) = (2\pi)^{-(n-1)/2}\frac{1}{\sigma^{2(n-1)/2}}|W|^{-1/2}\exp\left\{-\frac{1}{2\sigma^2}\underline{y}'AW^{-1}A'\underline{y}\right\}.$$

(d)

$$\log L(\sigma^2; A'\underline{y}) = -\frac{n-1}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\underline{y}'AW^{-1}A'\underline{y} + \text{constant}.$$
$$\frac{\partial \log L(\sigma^2; A'\underline{y})}{\partial\sigma^2} = -\frac{n-1}{2\sigma^2} + \frac{\underline{y}'AW^{-1}A'\underline{y}}{2\sigma^4}$$

Equating the derivative to 0 and solving for σ^2 yields $\hat{\sigma}^2 = \underline{y}'AW^{-1}A'\underline{y}/(n-1)$. Using Problem 9 from HW 3, we have $W^{-1} = I + \underline{1}\underline{1}'$. Thus

$$\begin{aligned}
\underline{y}'AW^{-1}A'\underline{y} &= [y_2 - \bar{y}, \ldots, y_n - \bar{y}](I + \underline{1}\underline{1}')[y_2 - \bar{y}, \ldots, y_n - \bar{y}]' \\
&= [y_2 - \bar{y}, \ldots, y_n - \bar{y}]I[y_2 - \bar{y}, \ldots, y_n - \bar{y}]' \\
&\quad + [y_2 - \bar{y}, \ldots, y_n - \bar{y}]\underline{1}\underline{1}'[y_2 - \bar{y}, \ldots, y_n - \bar{y}]' \\
&= \sum_{i=2}^n (y_i - \bar{y})^2 + \left\{\sum_{i=2}^n (y_i - \bar{y})\right\}^2 \\
&= \sum_{i=2}^n (y_i - \bar{y})^2 + \{-(y_1 - \bar{y})\}^2 \\
&= \sum_{i=1}^n (y_i - \bar{y})^2.
\end{aligned}$$

Thus $\hat{\sigma}^2 = \sum_{i=1}^n (y_i - \bar{y})^2/(n-1)$.

(e) The REML estimator is the same as the usual unbiased estimator.

(f) The REML estimator is invariant to the set of $n - p^*$ LIN independent error contrasts as demonstrated in class.