

1. Prove or disprove the following statement:

For any nonzero matrix X and any $(X'X)^-$, a generalized inverse of $X'X$,
 $(X'X)^- = [(X'X)^-]'$.

2. Suppose J is an $n \times n$ matrix consisting entirely of ones. Prove that G is a generalized inverse of J if and only if the sum of the elements of G is equal to a particular constant value k . Also, state the value of the constant k .
3. Show that if A and B are nonnegative definite matrices, then $\text{tr}(AB) \geq 0$.
4. Prove or disprove the following:

Suppose $\underline{y} \neq \underline{0}$ and $\underline{y} \in \mathcal{R}(A)$. Then \underline{w} is a solution to the linear equation $A\underline{x} = \underline{y}$ (i.e., $A\underline{w} = \underline{y}$) if and only if there exists G , a generalized inverse of A , such $\underline{w} = G\underline{y}$.

5. Suppose

$$\underline{y} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \text{ and } X = \begin{bmatrix} -1 & 1 & 5 \\ 1 & 0 & 7 \\ 0 & 0 & 0 \end{bmatrix}.$$

- (a) Sketch a three-dimensional plot that shows $\mathcal{R}(X)$, $\mathcal{R}(X)^\perp$, \underline{y} , $P_X \underline{y}$, and $\underline{y} - P_X \underline{y}$.
- (b) Determine the rank of $X'X$.
- (c) Use the following information to find a generalized inverse of $X'X$.

Let A be an $n \times m$ matrix of rank r . Let B be a nonsingular $r \times r$ submatrix of A . Let G denote the matrix formed as follows. Start with A and replace the entries of B in A with those of $(B^{-1})'$. Then replace all other entries of A with 0. Take the transpose of the resulting matrix to end up with G . Then G is a generalized inverse of A .

- (d) Compute a solution $\hat{\underline{\beta}}$ to the normal equations.
- (e) Check that the vectors $X\hat{\underline{\beta}}$ and $\underline{y} - X\hat{\underline{\beta}}$ agree with your results in part (a).
6. Consider the simple linear regression model $y_i = \beta_0 + \beta_1 x_i + e_i$ for $i = 1, \dots, n$. Let $\underline{y} = (y_1, \dots, y_n)'$ and $\underline{x} = (x_1, \dots, x_n)'$. Let $X = [\underline{1}, \underline{x}]$ and assume that $x_i \neq x_j$ for some $i \neq j$. Find the solution to the normal equations using matrix and vector notation and verify that the

solution is equivalent to the standard summation formulas for the least squares estimates of β_0 and β_1 presented in a first course on simple linear regression:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})y_i}{\sum_{i=1}^n (x_i - \bar{x})^2} \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}.$$

7. Partition \underline{y} as $\underline{y} = [\underline{y}'_1, \underline{y}'_2]'$ and take $X = [X'_1, X'_2]'$ and $\underline{e} = [\underline{e}'_1, \underline{e}'_2]'$ to be the corresponding partitionings of X and \underline{e} . Suppose that $\mathcal{R}(X'_1) \cap \mathcal{R}(X'_2) = \{0\}$. Show that if $\underline{\lambda} \in \mathcal{R}(X'_1)$, then the least squares estimate of $\underline{\lambda}'\underline{\beta}$ based on the model $\underline{y}_1 = X_1\underline{\beta}_1 + \underline{e}_1$ is the same as the least squares estimate of $\underline{\lambda}'\underline{\beta}$ based on the model $\underline{y} = X\underline{\beta} + \underline{e}$.

8. Let $M = \begin{bmatrix} A & C \\ D & B \end{bmatrix}$.

(a) If $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(D') \subseteq \mathcal{R}(A')$, then

$$M^- = \begin{bmatrix} A^- + A^-CQ^-DA^- & -A^-CQ^- \\ -Q^-DA^- & Q^- \end{bmatrix}$$

where $Q = B - DA^-C$.

(b) If $\mathcal{R}(D) \subseteq \mathcal{R}(B)$ and $\mathcal{R}(C') \subseteq \mathcal{R}(B')$, then

$$M^- = \begin{bmatrix} P^- & -P^-CB^- \\ -B^-DP^- & B^- + B^-DP^-CB^- \end{bmatrix}$$

where $P = A - CB^-D$.

Prove only part (b). A proof for part (a) is similar. One way to prove this is by direct multiplication of MM^-M to get M . However, that is not very fun and more challenging than you might expect. Another suggested proof is outlined as follows.

First prove the following lemma:

If $W = XYZ$ where X and Z are nonsingular, then $Z^{-1}Y^{-1}X^{-1}$ is a generalized inverse of W .

Next prove that

$$\begin{bmatrix} A & C \\ D & B \end{bmatrix} = \begin{bmatrix} I & CB^- \\ 0 & I \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I & 0 \\ B^-D & I \end{bmatrix}$$

and then apply the lemma.

9. Prove the following: Suppose $W = R + D$ where $\mathcal{R}(D) \subseteq \mathcal{R}(R)$ and $\mathcal{R}(D') \subseteq \mathcal{R}(R')$. If S , T , and U are any three matrices such that $STU = D$, then

$$W^- = R^- - R^-ST(T + TUR^-ST)^-TUR^-.$$