STAT611 **Homework 2** **Solutions**

1. Suppose $A : n \times p$, $B : m \times n$, $C : p \times q$, $\text{rank}(B) = n$, and $\text{rank}(C) = p$. Prove that $\text{rank}(A) = \text{rank}(BAC)$.

Let $r = \text{rank(A)}$. Then A has r linearly independent columns. Let $\underline{a}_{[1]}, \ldots, \underline{a}_{[r]}$ denote r linearly independent columns of A. Suppose

$$x_1 B\underline{a}_{[1]} + \cdots + x_r B\underline{a}_{[r]} = \underline{0}.$$

Then

$$B(x_1\underline{a}_{[1]} + \cdots + x_r\underline{a}_{[r]}) = \underline{0}$$

which implies

$$x_1\underline{a}_{[1]} + \cdots + x_r\underline{a}_{[r]} = \underline{0}$$

because the columns of B are linearly independent (rank(B) =number of columns of B). Now it must be that $\underline{x} = \underline{0}$ due to the linear independence of $\underline{a}_{[1]}, \ldots, \underline{a}_{[r]}$. This proves that $B\underline{a}_{[1]}, \ldots, B\underline{a}_{[r]}$ are linearly independent. Since these are r columns of BA, we have established that $\text{rank(BA)} \geq \text{r} = \text{rank(A)}$. By Theorem A.23 (iv), we have $\text{rank(BA)} \leq \text{rank(A)}$. Thus, we must have $\text{rank(BA)} = \text{rank(A)}$.

Now let $D = BA$. By the above work, we know that $\text{rank(D)} = \text{rank(A)} = \text{r}$. Thus, D has r linearly independent rows, say, $\underline{d}'_{[1]}, \ldots, \underline{d}'_{[r]}$. We now apply the same argument used previously:

$$\begin{aligned} & x_1 C'\underline{d}_{[1]} + \cdots + x_r C'\underline{d}_{[r]} = \underline{0} \\ \Rightarrow \quad & C'(x_1\underline{d}_{[1]} + \cdots + x_r\underline{d}_{[r]}) = \underline{0} \\ \Rightarrow \quad & x_1\underline{d}_{[1]} + \cdots + x_r\underline{d}_{[r]} = \underline{0} \text{ (by LI of rows of } C) \\ \Rightarrow \quad & \underline{x} = \underline{0} \text{ (by LI of } \underline{d}_{[1]}, \ldots, \underline{d}_{[r]}). \end{aligned}$$

Thus, $C'\underline{d}_{[1]}, \ldots, C'\underline{d}_{[r]}$ are linearly independent. Because these vectors are the rows of DC, we have $\text{rank(DC)} \geq \text{r} = \text{rank(D)}$. By Theorem A.23 (iv), we have $\text{rank(DC)} \leq \text{rank(D)} = \text{r}$. Thus, we must have $\text{rank(DC)} = \text{r}$. Because $DC = BAC$ and $r = \text{rank(A)}$, we have $\text{rank(BAC)} = \text{rank(A)}$.

Note that some students used the fact that $B'B$ and CC' are each nonsingular when B is of full column rank and C is of full row rank. Using these facts leads to a much shorter proof. However, we had not established those facts in class at the point that this question was posed in the notes. Thus, I offered the proof above which uses only results in notes that came prior to the statement of the problem. Student who used those facts should be sure that they can prove them and that their proof does not depend on the result they are trying to prove. In this case, there is no problem because it is easy to establish the nonsingularity of $B'B$ and CC' using facts about P_X, for instance.

2. Let A be an arbitrary $n \times p$ matrix of rank r. By the Singular Value Decomposition Theorem, we have $A = UDV'$ where $U : n \times n$ is orthogonal, $V : p \times p$ is orthogonal, and

$$D = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}$$

with $L = \text{diag}(\ell_1, \ldots, \ell_r)$ and $\ell_i > 0$ for all $i = 1, \ldots, r$.

(a) Show that the columns of U are eigenvectors of AA'.

First note that

$$AA' = UDV'VD'U' = UDD'U' \text{ and that } DD' = \text{diag}(\ell_1^2, \ldots, \ell_r^2, 0, \ldots, 0)$$

with $n - r$ trailing zeros so that DD' is $n \times n$.

Now note that

$$\begin{aligned} |AA' - \lambda I| &= |UDD'U' - \lambda I| \\ &= |UDD'U' - \lambda UU'| \\ &= |U(DD' - \lambda I)U'| \\ &= |U||DD' - \lambda I||U'| \\ &= |UU'||DD' - \lambda I| \\ &= |I||DD' - \lambda I| \\ &= |DD' - \lambda I| = \prod_{i=1}^{r} (\ell_i^2 - \lambda)(-\lambda)^{n-r}. \end{aligned}$$

Thus, the non zero eigenvalues of AA' are $\ell_1^2, \ldots, \ell_r^2$. Finally, note that

$$AA'\underline{u}_{(i)} = UDD'U'\underline{u}_{(i)} = UDD'\underline{e}_i = \ell_i^2 U\underline{e}_i = \ell_i^2 \underline{u}_{(i)} \ \forall \ i = 1, \ldots, r.$$

Thus, the columns of U are the eigenvectors of AA'.

(b) Show that the columns of V are eigenvectors of $A'A$.

The argument here is completely analogous to the argument used to prove part (a).

(c) Show that the nonzero eigenvalues of AA' are the same as the nonzero eigenvalues of $A'A$ and that these eigenvalues are $\ell_1^2, \ldots, \ell_r^2$.

We already showed that $\ell_1^2, \ldots, \ell_r^2$ are the nonzero eigenvalues of AA' in part (a). The same basic argument shows that these are also the nonzero eigenvalues of $A'A$.

3. Show that any matrix satisfying the Moore-Penrose conditions is unique.

Suppose G_1 and G_2 are any two matrices satisfying the four Moore-Penrose conditions. Then we have

$$
\begin{aligned}
G_1 &= G_1AG_1 = G_1AG_2AG_1 = G_1AG_2G_1'A' = G_1G_2'A'G_1'A' = G_1G_2'A' \\
&= G_1AG_2 = A'G_1'G_2 = A'G_1'G_2AG_2 = A'G_1'A'G_2'G_2 = A'G_2'G_2 \\
&= G_2AG_2 = G_2.
\end{aligned}
$$

Therefore, the Moore-Penrose generalized inverse is unique.

4. Let A be an arbitrary $n \times p$ matrix of rank r. By the Singular Value Decomposition Theorem, we have $A = ULV'$ where $U : n \times r$, $V : p \times r$, $U'U = V'V = I : r \times r$, $L = \text{diag}(\ell_1, \ldots, \ell_r)$ with $\ell_i > 0$ for all $i = 1, \ldots, r$. Is $VL^{-1}U'$ the Moore-Penrose inverse of A? Explain.

Yes. $VL^{-1}U'$ matrix satisfies the four conditions as follows:

(a) $AVL^{-1}U'A = ULV'VL^{-1}U'ULV' = ULL^{-1}U'ULV' = UU'ULV' = ULV' = A$

(b) $VL^{-1}U'AVL^{-1}U' = VL^{-1}U'ULV'VL^{-1}U' = VL^{-1}LV'VL^{-1}U' = VV'VL^{-1}U' = VL^{-1}U'$

(c) $VL^{-1}U'A = VL^{-1}U'ULV' = VL^{-1}LV' = VV'$ which is clearly symmetric.

(d) $AVL^{-1}U' = ULV'VL^{-1}U' = ULL^{-1}U' = UU'$ which is clearly symmetric.

5. Suppose A is an $m \times n$ matrix. Prove that all nonzero eigenvalues of $A'A$ are positive. Note that we used this lemma to prove the Spectral Decomposition Theorem. Thus, your proof should not use the Spectral Decomposition Theorem or any other facts that we proved using the Spectral Decomposition Theorem. Use only facts and results from your notes that were presented before the statement of this lemma.

Let $\lambda \neq 0$ be an eigenvalue of $A'A$. Then

$$\begin{aligned} |A'A - \lambda I| = 0 \quad &\Rightarrow \quad \exists\, \underline{x} \neq 0 \ni (A'A - \lambda I)\underline{x} = \underline{0} \\ &\Rightarrow \quad \exists\, \underline{x} \neq 0 \ni A'A\underline{x} = \lambda \underline{x} \\ &\Rightarrow \quad \exists\, \underline{x} \neq 0 \ni \underline{x}'A'A\underline{x} = \lambda \underline{x}'\underline{x} \\ &\Rightarrow \quad \exists\, \underline{x} \neq 0 \ni \lambda = \underline{x}'A'A\underline{x}/\underline{x}'\underline{x}. \end{aligned}$$

Now note $\underline{x}'A'A\underline{x}/\underline{x}'\underline{x} = ||\underline{y}||^2 \geq 0$ where $\underline{y} = A\underline{x}/||\underline{x}||$. Thus, $\lambda \geq 0$. It follows that $\lambda > 0$ because λ was assumed to be a nonzero eigenvalue of $A'A$.

6. Prove that all the eigenvalues of a symmetric matrix A are 0 if and only if $A = 0$. Use only facts and results from your notes that were presented before the statement of this lemma. You may wish to use the result from the previous problem in your proof. That is acceptable even if you weren't able to prove the previous result.

   If $A_{p\times p} = 0$, then $|A - \lambda I| = |-\lambda I| = (-\lambda)^p$. The roots of this polynomial are all zero. Thus, $A = 0$ implies all the eigenvalues of A are zero.

   To prove the converse, note the following:

$$\begin{aligned} &|A - \lambda I| = 0 \text{ if and only if } \lambda = 0 \\ \Rightarrow\ & |A - \lambda I||A - (-\lambda)I| = 0 \text{ iff } \lambda = 0 \\ \Rightarrow\ & |AA - \lambda^2 I| = 0 \text{ iff } \lambda = 0 \\ \Rightarrow\ & |A'A - \lambda^2 I| = 0 \text{ iff } \lambda = 0 \\ \Rightarrow\ & |A'A - \lambda I| = 0 \text{ iff } \lambda = 0 \text{ by the result proved in question 5} \\ \Rightarrow\ & \text{ all eigenvalues of } A'A \text{ are zero} \\ \Rightarrow\ & \sum_{i=1}^{p}\sum_{j=1}^{p} a_{ij}^2 = \mathrm{tr(A'A)} = \sum_{\mathrm{i=1}}^{\mathrm{p}} \lambda_{\mathrm{i}} = 0 \\ \Rightarrow\ & a_{ij}^2 = 0\ \forall\, i, j \\ \Rightarrow\ & A = 0. \end{aligned}$$

7. Show that $\exists$ a symmetric matrix $A : n \times n$ for any matrix $B : n \times n$ such that $\underline{x}'B\underline{x} = \underline{x}'A\underline{x}\ \forall\ \underline{x} \in \mathbb{R}^n$. (Note that this implies that we can restrict our attention to quadratic forms involving symmetric matrices.)

   Let $A = \frac{1}{2}B + \frac{1}{2}B'$. Then A is symmetric, and

$$\underline{x}'A\underline{x} = \underline{x}'\left(\frac{1}{2}B + \frac{1}{2}B'\right)\underline{x} = \frac{1}{2}\underline{x}'B\underline{x} + \frac{1}{2}\underline{x}'B'\underline{x} = \underline{x}'B\underline{x}$$

because $\underline{x}'B'\underline{x}$ is a scalar which implies $\underline{x}'B'\underline{x} = (\underline{x}'B'\underline{x})' = \underline{x}'B\underline{x}$.

8. Suppose $A : n \times n$ is positive definite. Prove the following:

   (a) All the eigenvalues of A are positive.

   We will actually prove more than required here. $A_{n\times n}$ symmetric $\Rightarrow A = \Gamma\Lambda\Gamma'$ in accordance with the Spectral Decomposition Theorem. Now note that $\forall\ \underline{x} \neq \underline{0}$,

$$\underline{x}'A\underline{x} = \underline{x}'\Gamma\Lambda\Gamma'\underline{x} = \underline{z}'\Lambda\underline{z},$$

   where $\underline{z} = \Gamma'\underline{x} \neq \underline{0}$ (because $\underline{x} \neq \underline{0}$ and Γ' is nonsingular). Likewise $\forall\ \underline{z} \neq \underline{0}$,

$$\underline{z}'\Lambda\underline{z} = \underline{z}'\Gamma'\Gamma\Lambda\Gamma'\Gamma\underline{z} = \underline{z}'\Gamma'A\Gamma\underline{z} = \underline{x}'A\underline{x},$$

   where $\underline{x} = \Gamma\underline{z} \neq \underline{0}$ (because $\underline{z} \neq \underline{0}$ and Γ is nonsingular). Thus

$$\{\underline{x}'A\underline{x} : \underline{x} \in I\!R^n, \underline{x} \neq \underline{0}\} = \{\underline{z}'\Lambda\underline{z} : \underline{z} \in I\!R^n, \underline{z} \neq \underline{0}\}. \tag{1}$$

   Now by taking $\underline{z}$ to be columns of $I_{n\times n}$, we see from (1) that

$$\lambda_1, \ldots \lambda_n \in \{\underline{x}'A\underline{x} : \underline{x} \in I\!R^n, \underline{x} \neq \underline{0}\}. \tag{2}$$

   Now by (2), A positive definite $\Rightarrow \lambda_i > 0$ for any $i = 1, \ldots, n$. This completes the proof of the homework problem, but we can go on to prove the converse. Namely, we can establish that if $A_{n\times n}$ is symmetric with all positive eigenvalues, then A is positive definite. By (1) we, need only argue that $\underline{z}'\Lambda\underline{z} = \sum_{i=1}^n \lambda_i z_i^2 > 0\ \forall\ \underline{z} \neq \underline{0}$. This is clearly true because $\lambda_i > 0$ for all i and $z_i^2 \geq 0$ for all i with strict inequality for at least one i.

   (b) $|A| > 0$.

   $|A| = \prod_{i=1}^n \lambda_i > 0$ by (a).

   (c) A^{-1} is positive definite.

   By the additional argument at the end of the proof of part (a), A^{-1} will be positive definite if all its eigenvalues are positive. Note that $A^{-1} = \Gamma\Lambda^{-1}\Gamma'$. Given this expression, we can show that the eigenvalues of A^{-1} are $1/\lambda_1, \ldots, 1/\lambda_n$ as follows:

$$|A^{-1} - \lambda I| \;=\; |\Gamma\Lambda^{-1}\Gamma' - \lambda I|$$

$$
\begin{aligned}
&= |\Gamma\Lambda^{-1}\Gamma' - \lambda\Gamma\Gamma'| \\
&= |\Gamma(\Lambda^{-1} - \lambda I)\Gamma'| \\
&= |\Gamma||\Lambda^{-1} - \lambda I||\Gamma'| \\
&= |\Gamma\Gamma'||\Lambda^{-1} - \lambda I| \\
&= |I||\Lambda^{-1} - \lambda I| \\
&= |\Lambda^{-1} - \lambda I|.
\end{aligned}
$$

Now note that $|\Lambda^{-1} - \lambda I| = \prod_{i=1}^{n}(1/\lambda_i - \lambda)$. Thus, the eigenvalues of A^{-1} are $1/\lambda_1, \ldots, 1/\lambda_n$. It follows that $1/\lambda_1, \ldots, 1/\lambda_n > 0$ because $\lambda_1, \ldots, \lambda_n > 0$.

(d) If $P : n \times m$ is of rank m, then $P'AP$ is positive definite.

Let $\underline{x}$ be an arbitrary nonzero vector. Now $\underline{x} \neq \underline{0}$ and P of full column rank implies $\underline{y} \equiv P\underline{x} \neq \underline{0}$. Now $\underline{y} \neq \underline{0}$ implies $\underline{x}'P'AP\underline{x} = \underline{y}'A\underline{y} > \underline{0}$ by positive definiteness of A. Thus, $\underline{x} \neq \underline{0} \Rightarrow \underline{x}'P'AP\underline{x} > 0$.

(e) If $P : n \times m$ is of rank $r < \min(m, n)$, then $P'AP$ is positive semi definite.

A positive definite $\Rightarrow \underline{x}'P'AP\underline{x} \geq 0$ with equality if and only if $P\underline{x} = \underline{0}$. Thus, $P'AP$ is nonnegative definite. Because P has rank less than m, the columns of P are linearly dependent. Thus, there exists $\underline{x} \neq \underline{0}$ such that $P\underline{x} = \underline{0}$. Thus, there exists $\underline{x} \neq \underline{0}$ such that $\underline{x}'P'AP\underline{x} = 0$. Therefore, $P'AP$ is positive semidefinite.

9. Prove that if A is idempotent, then $\mathrm{tr}(A) = \mathrm{rank}(A)$.

Suppose $A : m \times n$ has rank r. (Because A is idempotent, we know A must be square ($m = n$), but we will begin by establishing a result that holds for more $m \times n$ matrices in general.) By the Singular Value Decomposition Theorem, we have

$$
A = U_{m\times m} \begin{bmatrix} L_{r\times r} & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix} V'_{n\times n}
$$

where U and V are orthogonal and L is diagonal with positive diagonal elements $\ell_1, \ldots, \ell_r$. Let

$$
B = U \begin{bmatrix} L & 0_{r\times(m-r)} \\ 0_{(m-r)\times r} & I_{(m-r)\times(m-r)} \end{bmatrix} \text{ and } C = V'.
$$

Then

$$B \begin{bmatrix} I_{r\times r} & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix} C = U \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} V' = A.$$

Note that B and C are each nonsingular because

$$B^{-1} = \begin{bmatrix} L^{-1} & 0 \\ 0 & I \end{bmatrix} U' \text{ and } C^{-1} = V.$$

Now A idempotent implies

$$B \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} CB \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} C = B \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} C.$$

Left multiplying by B^{-1} and right multiplying by C^{-1} on both sides of the above equation yields

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} CB \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, we have

$$\begin{aligned} \text{rank(A)} &= r = \text{rank}\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right) = \text{tr}\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right) = \text{tr}\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \text{CB} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right) \\ &= \text{tr}\left(\text{B}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\text{C}\right) = \text{tr}\left(\text{B}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\text{C}\right) = \text{tr(A)}. \end{aligned}$$