

1. Suppose $A : n \times p$, $B : m \times n$, $C : p \times q$, $\text{rank}(B) = n$, and $\text{rank}(C) = p$. Prove that $\text{rank}(A) = \text{rank}(BAC)$.
2. Let A be an arbitrary $n \times p$ matrix of rank r . By the Singular Value Decomposition Theorem, we have $A = UDV'$ where $U : n \times n$ is orthogonal, $V : p \times p$ is orthogonal, and

$$D = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}$$

with $L = \text{diag}(\ell_1, \dots, \ell_r)$ and $\ell_i > 0$ for all $i = 1, \dots, r$.

- (a) Show that the columns of U are eigenvectors of AA' .
 - (b) Show that the columns of V are eigenvectors of $A'A$.
 - (c) Show that the nonzero eigenvalues of AA' are the same as the nonzero eigenvalues of $A'A$ and that these eigenvalues are $\ell_1^2, \dots, \ell_r^2$.
3. Show that any matrix satisfying the Moore-Penrose conditions is unique.
 4. Let A be an arbitrary $n \times p$ matrix of rank r . By the Singular Value Decomposition Theorem, we have $A = ULV'$ where $U : n \times r$, $V : p \times r$, $U'U = V'V = I : r \times r$, $L = \text{diag}(\ell_1, \dots, \ell_r)$ with $\ell_i > 0$ for all $i = 1, \dots, r$. Is $VL^{-1}U'$ the Moore-Penrose inverse of A ? Explain.
 5. Suppose A is an $m \times n$ matrix. Prove that all nonzero eigenvalues of $A'A$ are positive. Note that we used this lemma to prove the Spectral Decomposition Theorem. Thus, your proof should not use the Spectral Decomposition Theorem or any other facts that we proved using the Spectral Decomposition Theorem. Use only facts and results from your notes that were presented before the statement of this lemma.
 6. Prove that all the eigenvalues of a symmetric matrix A are 0 if and only if $A = 0$. Use only facts and results from your notes that were presented before the statement of this lemma. You may wish to use the result from the previous problem in your proof. That is acceptable even if you weren't able to prove the previous result.
 7. Show that \exists a symmetric matrix $A : n \times n$ for any matrix $B : n \times n$ such that $\underline{x}'B\underline{x} = \underline{x}'A\underline{x} \forall \underline{x} \in \mathbb{R}^n$. (Note that this implies that we can restrict our attention to quadratic forms involving symmetric matrices.)
 8. Suppose $A : n \times n$ is positive definite. Prove the following:

- (a) All the eigenvalues of A are positive.
- (b) $|A| > 0$.
- (c) A^{-1} is positive definite.
- (d) If $P : n \times m$ is of rank m , then $P'AP$ is positive definite.
- (e) If $P : n \times m$ is of rank $r < \min(m, n)$, then $P'AP$ is positive semi definite.

9. Prove that if A is idempotent, then $\text{tr}(A) = \text{rank}(A)$.