STAT611 **Homework 1** **Solutions**

1. Prove that $(AB)' = B'A'$ for matrices $A_{m\times n}$ and $B_{n\times p}$.

   The element in the jth row and ith column of AB is $\underline{a}_j'\underline{b}_{(i)}$. By definition of matrix transpose, the element in the ith row and jth column of $(AB)'$ is then $\underline{a}_j'\underline{b}_{(i)} = \underline{b}_{(i)}'\underline{a}_j$. Now

$$B'A' = \begin{bmatrix} \underline{b}_{(1)}' \\ \vdots \\ \underline{b}_{(p)}' \end{bmatrix} [\underline{a}_1, \ldots, \underline{a}_m].$$

   Thus the element in the ith row and j column of $B'A'$ is $\underline{b}_{(i)}'\underline{a}_j$.

2. Prove that the $\text{tr}(AB) = \text{tr}(BA)$ if both AB and BA are defined.

   Suppose $A_{m\times n}$ and $B_{n\times m}$. The diagonal elements of AB are $\underline{a}_1'\underline{b}_{(1)}, \ldots, \underline{a}_m'\underline{b}_{(m)}$. Note $\underline{a}_i'\underline{b}_{(i)} = \sum_{j=1}^n a_{ij}b_{ji}$. Thus,

$$\begin{aligned} \text{tr}(AB) &= \sum_{j=1}^m \underline{a}_j'\underline{b}_{(j)} \\ &= \sum_{j=1}^m \underline{b}_{(j)}'\underline{a}_j \\ &= \sum_{j=1}^m \sum_{i=1}^n b_{ij}a_{ji} \\ &= \sum_{i=1}^n \sum_{j=1}^m b_{ij}a_{ji} \\ &= \sum_{i=1}^n \underline{b}_i'\underline{a}_{(i)} \\ &= \text{tr(BA)}. \end{aligned}$$

3. Using the description of the determinant in Appendix A.3 of the text, prove the following:

   (a) The determinant of a 3×3 matrix is

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}.$$

   Expanding on the first row, the determinant of a 2×2 matrix is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a(-1)^{1+1}d + b(-1)^{1+2}c = ad - bc.$$

Thus, expanding on the first row of a 3×3 matrix, we have

$$\begin{aligned} |A_{3\times 3}| &= a_{11}(-1)^{1+1}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(-1)^{1+2}(a_{21}a_{33} - a_{23}a_{31}) \\ &\quad + a_{13}(-1)^{1+3}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}. \end{aligned}$$

(b) $|A| = |A'|$ for a square matrix A.

Suppose there exists an integer $n \geq 1$ such that $|C| = |C'|$ for any matrix C of dimensions $n \times n$. Let A denote an arbitrary matrix of dimension $(n+1) \times (n+1)$. Let $B = [b_{ij}] = A'$, and let N_{ij} denote the matrix obtained by deleting the ith row and jth column of B. Then

$$\begin{aligned} |A| &= \sum_{i=1}^{n+1} a_{i1}(-1)^{i+1}|M_{i1}| \\ &= \sum_{i=1}^{n+1} a_{i1}(-1)^{i+1}|M'_{i1}| \text{ (because } M_{i1} \text{ is } n \times n) \\ &= \sum_{i=1}^{n+1} b_{1i}(-1)^{1+i}|N_{1i}| \\ &= |B| = |A'|. \end{aligned}$$

Now clearly the determinant of any 1×1 matrix C satisfies $|C| = |C'|$. Hence, by induction, the result follows.

(c) The determinant of an upper triangular matrix is the product of its diagonal elements.

Suppose there exists an integer $n \geq 1$ such that $|C| = \prod_{i=1}^{n} c_{ii}$ for any upper triangular matrix C of dimensions $n \times n$. Let A denote an arbitrary upper triangular matrix of dimension $(n+1) \times (n+1)$. Then

$$\begin{aligned} |A| &= \sum_{i=1}^{n+1} a_{i1}(-1)^{i+1}|M_{i1}| \\ &= a_{11}(-1)^{1+1}|M_{11}| + \sum_{i=2}^{n+1} 0(-1)^{i+1}|M_{i1}| \\ &= a_{11}(-1)^{1+1}|M_{11}| \\ &= a_{11}\prod_{i=2}^{n+1} a_{ii} \\ &= \prod_{i=1}^{n+1} a_{ii}, \end{aligned}$$

where the second to last equality follows because M_{11} is upper triangular of dimension $n \times n$ with diagonal elements $a_{22}, \ldots, a_{n+1,n+1}$. Now clearly the determinant of any 1×1 or 2×2 upper triangular matrix is equal to the product of its diagonal elements. Hence, by induction, the result follows.

4. As discussed in class, the determinant of an $n \times n$ matrix can be defined as a sum of $n!$ products, where each product is plus or minus the product of n elements from the matrix with exactly one element from each row and one element from each column. Consider the matrix

$$\begin{bmatrix} 4 & -5 & 7 & 2 \\ -3 & 6 & -1 & 9 \\ 8 & 0 & 1 & 5 \\ -2 & 3 & -6 & -7 \end{bmatrix}.$$

List all products that begin with the element 7 and determine the sign attached to each product.

$$\begin{array}{ll} + & (7)(-3)(0)(-7) \\ - & (7)(-3)(5)(3) \\ - & (7)(6)(8)(-7) \\ + & (7)(6)(5)(-2) \\ + & (7)(9)(8)(3) \\ - & (7)(9)(0)(-2) \end{array}$$

5. Prove that the determinant of an orthogonal matrix is either 1 or -1.

$$1 = |I| = |P'P| = |P'||P| = |P||P| = |P|^2 \Rightarrow |P| \pm 1.$$

6. Let

$$\underline{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \underline{a}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \underline{a}_3 = \begin{bmatrix} 0 \\ -6 \\ 2 \\ 0 \end{bmatrix}, \text{ and } \underline{a}_4 = \begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}.$$

(a) Show that $\underline{a}_1, \ldots, \underline{a}_4$ are linearly dependent.

Note that

$$4\underline{a}_1 - 2\underline{a}_2 = \underline{a}_3.$$

Thus, by V.1 in our notes, $\underline{a}_1, \ldots, \underline{a}_4$ are linearly dependent.

(b) Is the vector $[1, 1, 0, 1]'$ in the vector space spanned by $\underline{a}_1, \ldots, \underline{a}_4$? Explain.

No.

$x_1\underline{a}_1+x_2\underline{a}_2+x_3\underline{a}_3+x_4\underline{a}_4 = [x_2+2x_2, -x_1+x_2-6x_3-3x_4, x_1+x_2+2x_3+x_4, 0]' \neq [1, 1, 0, 1]'$

for any $\underline{x} \in \mathbb{R}^4$ because 4th components differ regardless of $\underline{x} \in \mathbb{R}^4$.

(c) What is the dimension of the vector space spanned by $\underline{a}_1, \ldots, \underline{a}_4$?

The dimension is 2 because $\underline{a}_1$ and $\underline{a}_2$ are linearly independent vectors in the vector space and also span the vector space.

(d) Find two different bases for the vector space spanned by $\underline{a}_1, \ldots, \underline{a}_4$ such that no vector in one basis is a multiple of a vector in the other.

One basis is $\underline{a}_1$, $\underline{a}_2$. A second basis is $\underline{a}_1 + \underline{a}_2$, $\underline{a}_1 - \underline{a}_2$.

7. Prove that $\mathrm{rank(A + B)} \leq \mathrm{rank(A)} + \mathrm{rank(B)}$ for $m \times n$ matrices A and B.

$$\begin{aligned}
\mathrm{rank(A+B)} &= \mathrm{rank}\left([\mathrm{I},\mathrm{I}]\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} I \\ I \end{bmatrix}\right) \\
&\leq \mathrm{rank}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \quad \text{by A.23 (iv)} \\
&= \mathrm{rank(A)} + \mathrm{rank(B)}.
\end{aligned}$$

8. Prove the following:

(a) If $\underline{a}_1, \ldots, \underline{a}_p \in \mathbb{R}^n$ are orthonormal vectors (i.e., $||\underline{a}_i|| = 1 \; \forall \; i$ and $\underline{a}_i'\underline{a}_j = 0 \; \forall \; i \neq j$), then $\underline{a}_1, \ldots, \underline{a}_p$ are linearly independent.

Suppose $x_1\underline{a}_1 + \cdots + x_p\underline{a}_p = \underline{0}$. Then $\forall \; j = 1, \ldots, p$ we have

$$\begin{aligned}
&\underline{a}_j'(x_1\underline{a}_1 + \cdots + x_p\underline{a}_p) = \underline{a}_j'\underline{0} = 0 \; \forall \; j = 1, \ldots, p \\
\Rightarrow \;& x_1\underline{a}_j'\underline{a}_1 + \cdots + x_p\underline{a}_j'\underline{a}_p = 0 \; \forall \; j = 1, \ldots, p \\
\Rightarrow \;& x_j\underline{a}_j'\underline{a}_j = 0 \; \forall \; j = 1, \ldots, p \\
\Rightarrow \;& x_j = 0 \; \forall \; j = 1, \ldots, p.
\end{aligned}$$

Thus, $x_1\underline{a}_1 + \cdots + x_p\underline{a}_p = \underline{0}$ implies that $\underline{x} = \underline{0}$ and linear independence follows.

(b) If $\underline{a}_1, \ldots, \underline{a}_p \in I\!R^n$ are orthonormal vectors with $p < n$, then there exist vectors $\underline{a}_{p+1}, \ldots, \underline{a}_n$ such that $\underline{a}_1, \ldots, \underline{a}_n$ are orthonormal.

Let $k < n$ be arbitrary. Suppose the vectors $\underline{b}_1, \ldots, \underline{b}_k \in I\!R^n$ are orthonormal. The rows of the matrix $[\underline{b}_1, \ldots, \underline{b}_k]$ are row vectors with k components. Because there are n such rows and $n > k$, these rows must be linearly dependent by V.2 in our notes. Therefore, there exists a nonzero vector $\underline{x} \in I\!R^n$ such that $\underline{x}'[\underline{b}_1, \ldots, \underline{b}_k] = \underline{0}'$ or, equivalently, $\underline{x}'\underline{b}_j = 0 \ \forall \ j = 1, \ldots, k$. Let $\underline{b}_{k+1} = \underline{x}/||\underline{x}||$. Then $\underline{b}_1, \ldots, \underline{b}_{k+1}$ are orthonormal vectors in $I\!R^n$. Thus, starting with $\underline{a}_1, \ldots, \underline{a}_p$; we can apply this argument for $k = p$, then $k = p + 1$, etc. up to $k = n - 1$ to obtain $\underline{a}_1, \ldots, \underline{a}_n$ orthonormal.