

1. Prove that $(AB)' = B'A'$ for matrices $A_{m \times n}$ and $B_{n \times p}$.
2. Prove that the $\text{tr}(AB) = \text{tr}(BA)$ if both AB and BA are defined.
3. Using the description of the determinant in Appendix A.3 of the text, prove the following:

(a) The determinant of a 3×3 matrix is

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}.$$

(b) $|A| = |A'|$ for a square matrix A .

(c) The determinant of an upper triangular matrix is the product of its diagonal elements.

4. As discussed in class, the determinant of an $n \times n$ matrix can be defined as a sum of $n!$ products, where each product is plus or minus the product of n elements from the matrix with exactly one element from each row and one element from each column. Consider the matrix

$$\begin{bmatrix} 4 & -5 & 7 & 2 \\ -3 & 6 & -1 & 9 \\ 8 & 0 & 1 & 5 \\ -2 & 3 & -6 & -7 \end{bmatrix}.$$

List all products that begin with the element 7 and determine the sign attached to each product.

5. Prove that the determinant of an orthogonal matrix is either 1 or -1.

6. Let

$$\underline{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \underline{a}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \underline{a}_3 = \begin{bmatrix} 0 \\ -6 \\ 2 \\ 0 \end{bmatrix}, \text{ and } \underline{a}_4 = \begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}.$$

- (a) Show that $\underline{a}_1, \dots, \underline{a}_4$ are linearly dependent.
 - (b) Is the vector $[1, 1, 0, 1]'$ in the vector space spanned by $\underline{a}_1, \dots, \underline{a}_4$? Explain.
 - (c) What is the dimension of the vector space spanned by $\underline{a}_1, \dots, \underline{a}_4$?
 - (d) Find two different bases for the vector space spanned by $\underline{a}_1, \dots, \underline{a}_4$ such that no vector in one basis is a multiple of a vector in the other.
7. Prove that $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$ for $m \times n$ matrices A and B .

8. Prove the following:

- (a) If $\underline{a}_1, \dots, \underline{a}_p \in \mathbb{R}^n$ are orthonormal vectors (i.e., $\|\underline{a}_i\| = 1 \forall i$ and $\underline{a}_i' \underline{a}_j = 0 \forall i \neq j$), then $\underline{a}_1, \dots, \underline{a}_p$ are linearly independent.
- (b) If $\underline{a}_1, \dots, \underline{a}_p \in \mathbb{R}^n$ are orthonormal vectors with $p < n$, then there exist vectors $\underline{a}_{p+1}, \dots, \underline{a}_n$ such that $\underline{a}_1, \dots, \underline{a}_n$ are orthonormal.