STAT611 **Exam 2** **November 8, 2007**

1. Suppose $\underline{y} = X\underline{\beta} + \underline{\varepsilon}$, where $\underline{\varepsilon}$ has mean $\underline{0}$ and dispersion $\sigma^2 I$. Suppose $L'\underline{\beta}$ is estimable. Prove that the ordinary least squares estimator of $L'\underline{\beta}$ is $W'X'\underline{y}$ if and only if $X'XW = L$.

   Proof: $L'\underline{\beta}$ estimable implies there exists A such that $L' = A'X$. If $W'X'\underline{y}$ is the ordinary least squares estimator of $L'\underline{\beta}$, then

$$
\begin{aligned}
W'X'\underline{y} = L'(X'X)^{-}X'\underline{y} \ \forall \ \underline{y} \quad &\Rightarrow \quad W'X' = L'(X'X)^{-}X' \\
&\Rightarrow \quad W'X'X = L'(X'X)^{-}X'X \\
&\Rightarrow \quad W'X'X = A'X(X'X)^{-}X'X \\
&\Rightarrow \quad W'X'X = A'X \\
&\Rightarrow \quad W'X'X = L' \\
&\Rightarrow \quad X'XW = L.
\end{aligned}
$$

   Conversely,

$$
\begin{aligned}
X'XW = L \quad &\Rightarrow \quad W'X'X = L' \\
&\Rightarrow \quad W'X'X(X'X)^{-}X' = L'(X'X)^{-}X' \\
&\Rightarrow \quad W'X' = L'(X'X)^{-}X' \\
&\Rightarrow \quad W'X'\underline{y} = L'(X'X)^{-}X'\underline{y} \ \forall \ \underline{y} \\
&\Rightarrow \quad W'X'\underline{y} = L'\underline{\hat{\beta}}.
\end{aligned}
$$

2. Consider a completely randomized experiment with two factors denoted A and B. Suppose each factor has two levels and that the following data were collected.

Level of Factor A	Level of Factor B	Response
1	1	8
1	1	6
1	2	5
1	2	3
2	2	0
2	2	4

   Suppose that the additive model $y_{ijk} = \alpha_i + \beta_j + \varepsilon_{ijk}$ is appropriate, where y_{ijk} denotes the k^{th} observation for level i of factor A and level j of factor B ($i = 1, 2; j = 1, 2; k = 1, 2$) and the ε_{ijk} errors are independent and identically distributed as $N(0, \sigma^2)$ for some $\sigma^2 > 0$. Suppose we write this model as $\underline{y} = X\underline{\beta} + \underline{\varepsilon}$, where $\underline{y} = (8, 6, 5, 3, 0, 4)'$ and $\underline{\beta} = (\alpha_1, \alpha_2, \beta_1, \beta_2)'$. Here

α_1 and α_2 denote effects associated with levels 1 and 2 of factor A, respectively; and β_1 and β_2 denote effects associated with levels 1 and 2 of factor B, respectively.

> *Note that it is possible to solve all of the following problems without computing a generalized inverse of $X'X$. You may choose to compute a generalized inverse of $X'X$ by hand and solve the problems using the generalized inverse, but doing so will take several minutes and will require knowledge of one or more theorems from the notes and/or homework.*

(a) Provide the matrix X corresponding to the choice of $\underline{y}$ and $\underline{\beta}$ presented above.

$$X = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

(b) Note that there are no observations for level 2 of factor A and level 1 of factor B. Is $\alpha_2+\beta_1$ estimable? Demonstrate why or why not.

$$[1, 0, -1, 0, 1, 0]X\underline{\beta} = \alpha_2 + \beta_1.$$

Thus, $\alpha_2 + \beta_1$ is estimable.

(c) Determine $\hat{\underline{\beta}}$ so that $||\underline{y} - X\hat{\underline{\beta}}|| \leq ||\underline{y} - X\underline{\beta}||$ for all $\underline{\beta} \in \mathbb{R}^4$.

The mean responses are

A	B	Mean
1	1	7
1	2	4
2	2	2

Thus, the sum of squares will be minimized if $\hat{\alpha}_1 + \hat{\beta}_1 = 7$, $\hat{\alpha}_1 + \hat{\beta}_2 = 4$, and $\hat{\alpha}_2 + \hat{\beta}_2 = 2$. There are an infinite number of solutions to this system of equations. One solution is obtained by choosing $\hat{\alpha}_1 = 7$, $\hat{\beta}_1 = 0$, $\hat{\beta}_2 = -3$, and $\hat{\alpha}_2 = 5$.

(d) Provide an unbiased estimate of σ^2.

The sum of the squared errors is $1+1+1+1+4+4=12$. The rank of X is 3 because rows 1, 3, and 5 are linearly independent. Thus, $\hat{\sigma}^2 = 12/(6-3) = 4$.

(e) Determine the minimum variance unbiased estimate of $\alpha_1 - \alpha_2$.

$\hat{\alpha}_1 - \hat{\alpha}_2 = 7 - 5 = 2$

3. Suppose $\underline{x} \sim N_p(\underline{\mu}, \Sigma_{p\times p})$, Σ is nonsingular, and A is a symmetric $p \times p$ matrix. Prove that $\underline{x}'A\underline{x} \sim \chi^2_q(\underline{\mu}'A\underline{\mu})$ if $A\Sigma$ is idempotent and $\text{rank}(A) = q$.

By the Spectral Decomposition Theorem, there exists a symmetric, nonsingular matrix $\Sigma^{1/2}$ such that $\Sigma^{1/2}\Sigma^{1/2} = \Sigma$. Note that $\Sigma^{-1/2}\underline{x} \sim N_p(\Sigma^{-1/2}\underline{\mu}, I)$. By Theorem A.87,

$$\begin{aligned}\underline{x}'A\underline{x} = (\Sigma^{-1/2}\underline{x})'\Sigma^{1/2}A\Sigma^{1/2}(\Sigma^{-1/2}\underline{x}) &\sim \chi^2_{\text{rank}(\Sigma^{1/2}A\Sigma^{1/2})}[(\Sigma^{-1/2}\underline{\mu})'\Sigma^{1/2}A\Sigma^{1/2}(\Sigma^{-1/2}\underline{\mu})] \\ &= \chi^2_q[\underline{\mu}'A\underline{\mu}],\end{aligned}$$

provided that $\Sigma^{1/2}A\Sigma^{1/2}$ is idempotent. To see that this matrix is idempotent, note that

$$\begin{aligned}A\Sigma A\Sigma = A\Sigma &\Rightarrow A\Sigma^{1/2}\Sigma^{1/2}A\Sigma^{1/2}\Sigma^{1/2} = A\Sigma^{1/2}\Sigma^{1/2} \\ &\Rightarrow \Sigma^{1/2}A\Sigma^{1/2}\Sigma^{1/2}A\Sigma^{1/2}\Sigma^{1/2} = \Sigma^{1/2}A\Sigma^{1/2}\Sigma^{1/2} \\ &\Rightarrow \Sigma^{1/2}A\Sigma^{1/2}\Sigma^{1/2}A\Sigma^{1/2}\Sigma^{1/2}\Sigma^{-1/2} = \Sigma^{1/2}A\Sigma^{1/2}\Sigma^{1/2}\Sigma^{-1/2} \\ &\Rightarrow \Sigma^{1/2}A\Sigma^{1/2}\Sigma^{1/2}A\Sigma^{1/2} = \Sigma^{1/2}A\Sigma^{1/2}.\end{aligned}$$

4. Consider a balanced and completely randomized experiment with K treatments and n observations per treatment. Suppose the following linear model is appropriate.

$$y_{ij} = \mu_i + \varepsilon_{ij} \quad i = 1, \ldots, K \quad j = 1, \ldots, n$$

where $\mu_1, \ldots, \mu_K$ denote unknown treatment means and the errors ε_{ij} are independently distributed as $N(0, \sigma^2)$ for some unknown $\sigma^2 > 0$. Suppose we wish to test $H_{0i} : \mu_i = 0$ vs. $H_{Ai} : \mu_i \neq 0$ for each $i = 1, \ldots, K$.

(a) Write down the test statistic you will use to test H_{0i} vs. H_{Ai} and state its distribution precisely.

Because all observations are independent and each treatment has its own variance, we have K separate linear models. Each model has the form $\underline{y}_i = \underline{1}\mu_i + \underline{\varepsilon}_i$, where $\underline{y}_i = [y_{i1}, \ldots, y_{in}]'$ and $\underline{\varepsilon}_i = [\varepsilon_{i1}, \ldots, \varepsilon_{in}]'$. Our general result

$$\frac{(L'\hat{\underline{\beta}})'(L'(X'X)^-L)^{-1}L'\hat{\underline{\beta}}}{q\hat{\sigma}^2} \sim F_{q,T-K^*}[(L'\underline{\beta})'(L'(X'X)^-L)^{-1}L'\underline{\beta}/\sigma^2]$$

simplifies to

$$\frac{n\bar{y}_{i\cdot}^2}{\hat{\sigma}_i^2} \sim F_{1,n-1}[n\mu_i^2/\sigma_i^2],$$

where $\hat{\sigma}_i^2 = \sum_{j=1}^n (y_{ij} - \bar{y}_{i\cdot})^2/(n-1)$ for each $i = 1, \dots, K$. This is just the square of the simple one-sample t-statistic for testing whether a mean is 0.

(b) Suppose you want the probability of one or more type I errors in the family of K tests to be no larger than α in general, and to be exactly α if all K null hypotheses are true. Provide a multiple testing procedure that will satisfy these requirements.

Suppose each test is conducted at significance level α^*. Then, the probability that a type I error occurs for test i is α^* if H_{0i} is true and 0 otherwise. Then, due to independence of the tests, we have

$$\text{FWER} = 1 - P(\text{no type I errors}) \leq 1 - (1-\alpha^*)^K$$

with equality if and only if all K null hypotheses are true. Thus, the requirements of the problem will be satisfied if

$$\alpha = 1 - (1-\alpha^*)^K \iff \alpha^* = 1 - (1-\alpha)^{1/K}.$$

Thus, our testing procedure call for rejection of H_{0i} if and only if $\frac{n\bar{y}_{i\cdot}^2}{\hat{\sigma}_i^2} \geq F_{1,n-1,1-\alpha^*}$.