STAT611 **Exam 2** **November 8, 2007**

1. Suppose $\underline{y} = X\underline{\beta} + \underline{\varepsilon}$, where $\underline{\varepsilon}$ has mean $\underline{0}$ and dispersion $\sigma^2 I$. Suppose $L'\underline{\beta}$ is estimable. Prove that the ordinary least squares estimator of $L'\underline{\beta}$ is $W'X'\underline{y}$ if and only if $X'XW = L$. *(18 points)*

2. Consider a completely randomized experiment with two factors denoted A and B. Suppose each factor has two levels and that the following data were collected.

Level of Factor A	Level of Factor B	Response
1	1	8
1	1	6
1	2	5
1	2	3
2	2	0
2	2	4

Suppose that the additive model $y_{ijk} = \alpha_i + \beta_j + \varepsilon_{ijk}$ is appropriate, where y_{ijk} denotes the k^{th} observation for level i of factor A and level j of factor B ($i = 1, 2; j = 1, 2; k = 1, 2$) and the ε_{ijk} errors are independent and identically distributed as $N(0, \sigma^2)$ for some $\sigma^2 > 0$. Suppose we write this model as $\underline{y} = X\underline{\beta} + \underline{\varepsilon}$, where $\underline{y} = (8, 6, 5, 3, 0, 4)'$ and $\underline{\beta} = (\alpha_1, \alpha_2, \beta_1, \beta_2)'$. Here α_1 and α_2 denote effects associated with levels 1 and 2 of factor A, respectively; and β_1 and β_2 denote effects associated with levels 1 and 2 of factor B, respectively.

Note that it is possible to solve all of the following problems without computing a generalized inverse of $X'X$. You may choose to compute a generalized inverse of $X'X$ by hand and solve the problems using the generalized inverse, but doing so will take several minutes and will require knowledge of one or more theorems from the notes and/or homework.

(a) Provide the matrix X corresponding to the choice of $\underline{y}$ and $\underline{\beta}$ presented above. *(5 points)*

(b) Note that there are no observations for level 2 of factor A and level 1 of factor B. Is $\alpha_2+\beta_1$ estimable? Demonstrate why or why not. *(10 points)*

(c) Determine $\underline{\hat{\beta}}$ so that $||\underline{y} - X\underline{\hat{\beta}}|| \leq ||\underline{y} - X\underline{\beta}||$ for all $\underline{\beta} \in \mathbb{R}^4$. *(13 points)*

(d) Provide an unbiased estimate of σ^2. *(7 points)*

(e) Determine the minimum variance unbiased estimate of $\alpha_1 - \alpha_2$. *(5 points)*

3. Suppose $\underline{x} \sim N_p(\underline{\mu}, \Sigma_{p \times p})$, Σ is nonsingular, and A is a symmetric $p \times p$ matrix. Prove that $\underline{x}' A \underline{x} \sim \chi^2_q(\underline{\mu}' A \underline{\mu})$ if $A\Sigma$ is idempotent and $\text{rank}(A) = q$. *(20 points)*

4. Consider a balanced and completely randomized experiment with K treatments and n observations per treatment. Let y_{ij} denote the j^{th} observation for treatment i ($i = 1, \ldots, K$ and $j = 1, \ldots, n$). Suppose all responses are independent and that

$$y_{i1}, \ldots, y_{in} \overset{i.i.d.}{\sim} N(\mu_i, \sigma_i^2)$$

where μ_i is an unknown treatment mean and σ_i^2 is an unknown positive treatment variance for $i = 1, \ldots, K$. Suppose we wish to test $H_{0i} : \mu_i = 0$ vs. $H_{Ai} : \mu_i \neq 0$ for each $i = 1, \ldots, K$.

(a) Write down the test statistic you will use to test H_{0i} vs. H_{Ai}, and state its distribution precisely. *(10 points)*

(b) Suppose you want the probability of one or more type I errors in the family of K tests to be no larger than α in general and to be exactly α if all K null hypotheses are true. Provide a multiple testing procedure that will satisfy these requirements. *(12 points)*

Ungraded Scratch Paper

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