STAT611 **Exam 2** **November 6, 2012**

1. Suppose $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $\boldsymbol{y}$ is a vector of responses, $\boldsymbol{X}$ is a known and fixed $n \times p$ design matrix, $\boldsymbol{\beta}$ is an unknown parameter vector in $\mathbb{R}^p$, and $\boldsymbol{\epsilon}$ is a random and unobserved error vector.

   (a) What additional assumptions are necessary for the Aitken Model to hold?

   (b) Suppose the Aitken Model holds. Prove that $\boldsymbol{w}'\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y}$ is the BLUE of an estimable function $\boldsymbol{c}'\boldsymbol{\beta}$ if and only if $\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X}\boldsymbol{w} = \boldsymbol{c}$.

2. Suppose $Z_1, Z_2, Z_3 \overset{iid}{\sim} N(0,1)$. Derive the distribution of

$$W = \frac{2(Z_1 + Z_2 + Z_3)^2 + 3(Z_1 - Z_2)^2 + 12(Z_1 - Z_2) + 12}{2(Z_1 + Z_2 - 2Z_3)^2}.$$

3. Suppose $y_1, \ldots, y_{10} \overset{iid}{\sim} N(-1, \sigma^2)$ independent of $y_{11}, \ldots, y_{20} \overset{iid}{\sim} N(1, \sigma^2)$. Find the expected value of

$$s^2 = \frac{1}{19}\sum_{i=1}^{20}(y_i - \bar{y}_\cdot)^2.$$

4. Suppose $\boldsymbol{A}$ is an $n \times r$ matrix of rank r.

   (a) Prove that the eigenvalues of $\boldsymbol{A}'\boldsymbol{A}$ are the same as the non-zero eigenvalues of $\boldsymbol{A}\boldsymbol{A}'$.

   (b) Prove that $\boldsymbol{A} = \sum_{i=1}^{r}\sqrt{\lambda_i}u_i v_i'$, where $\lambda_1, \ldots, \lambda_r$ are the eigenvalues of $\boldsymbol{A}'\boldsymbol{A}$, $\boldsymbol{v}_i$ is a unit-norm eigenvector of $\boldsymbol{A}'\boldsymbol{A}$ corresponding to λ_i, and $\boldsymbol{u}_i$ is a unit-norm eigenvector of $\boldsymbol{A}\boldsymbol{A}'$ corresponding to λ_i for all $i = 1, \ldots, r$.

5. Suppose two treatments are randomly assigned to sows (mother pigs) using a completely randomized design. Let n_i denote the number of sows receiving treatment i for $i = 1, 2$. Suppose that the jth sow receiving treatment i has m_{ij} piglets (baby pigs), and let y_{ijk} denote the weight of piglet k born to sow j in treatment group i ($i = 1, 2; j = 1, \dots, n_i; k = 1, \dots, m_{ij}$). Consider the model

$$y_{ijk} = \mu_i + s_{ij} + e_{ijk},$$

where μ_1 and μ_2 are real-valued parameters, $s_{ij} \sim N(0, \sigma_s^2)$, and $e_{ijk} \sim N(0, \sigma_e^2)$ for all $i = 1, 2; j = 1, \dots, n_i; k = 1, \dots, m_{ij}$. Furthermore, suppose all s_{ij} and e_{ijk} terms are mutually independent, and suppose σ_s^2 and σ_e^2 are positive variance components satisfying $\sigma_s^2/\sigma_e^2 = 3$. Let

$$y_{i\cdot\cdot} = \sum_{j=1}^{n_i} \sum_{k=1}^{m_{ij}} y_{ijk} \text{ and } m_{i\cdot} = \sum_{j=1}^{n_i} m_{ij} \text{ for } i = 1, 2.$$

Fill in the blank in the statement below with a condition that is straightforward to understand and easy to check. (Although you do not need to prove that your answer is correct, you may wish to show your work or explain your reasoning so that you can receive partial credit if your answer is not right.)

The BLUE of $\mu_1 - \mu_2$ is $\dfrac{y_{1\cdot\cdot}}{m_{1\cdot}} - \dfrac{y_{2\cdot\cdot}}{m_{2\cdot}}$ if and only if ____.