STAT611 **Exam 1** **September 25, 2012**

1. Consider the general linear model discussed in class, where $\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{\epsilon}$ with $E(\boldsymbol{\epsilon}) = \boldsymbol{0}$.

   (a) State the definition of a linearly estimable function $\boldsymbol{c}'\boldsymbol{\beta}$.

   (b) State the definition of a least squares estimator of a linearly estimable function $\boldsymbol{c}'\boldsymbol{\beta}$.

   (c) State the normal equations.

   (d) Prove that if $\boldsymbol{c}'\hat{\boldsymbol{\beta}}$ is the same for all solutions $\hat{\boldsymbol{\beta}}$ to the normal equations, then $\boldsymbol{c}'\boldsymbol{\beta}$ is linearly estimable.

2. Suppose $\boldsymbol{A}$ is an $m \times n$ matrix. Prove that any matrix $\boldsymbol{G}$ satisfying $\boldsymbol{A}'\boldsymbol{A}\boldsymbol{G} = \boldsymbol{A}'$ is a generalized inverse of $\boldsymbol{A}$.

3. Suppose $\boldsymbol{A}$ is an $m \times n$ matrix and $\boldsymbol{B}$ is an $n \times r$ matrix. Prove that if $rank(\boldsymbol{A}) = n$, then $\boldsymbol{B}^{-}\boldsymbol{A}^{-}$ is a generalized inverse of $\boldsymbol{AB}$.

4. Is the following statement true or false?

   If $\mathcal{S}$ and $\mathcal{T}$ are vector spaces in $\mathbb{R}^n$ with $dim(\mathcal{S}) < dim(\mathcal{T})$, then there always exists a nonzero vector $\boldsymbol{x} \in \mathcal{T}$ such that $\boldsymbol{x} \in \mathcal{S}^{\perp}$.

   (You are only required to answer either true or false.)

5. Suppose two fertilizers (1 and 2) were randomly assigned to six pots, with three pots for each fertilizer. Suppose the three pots that received fertilizer 1 were randomly assigned to receive 1, 2, or 3 units of fertilizer 1. Likewise, suppose the three pots that received fertilizer 2 were randomly assigned to receive 1, 2, or 3 units of fertilizer 2. Each pot contained a seedling, and the height of each seedling was recorded six weeks after the application of fertilizer. For $i = 1, 2$ and $j = 1, 2, 3$, let y_{ij} denote the height of the seedling in the jth pot that received fertilizer i, and let x_{ij} denote the amount of fertilizer applied to the jth pot that received fertilizer i. Suppose the data are as follows.

Pot	i	j	Fertilizer	Amount of Fertilizer (x_{ij})	Height of Seedling (y_{ij})
1	1	1	1	1	8
2	1	2	1	2	12
3	1	3	1	3	14
4	2	1	2	1	6
5	2	2	2	2	15
6	2	3	2	3	17

Consider the general linear model given by

$$E(y_{ij}) = \mu + \phi_i + \gamma_i x_{ij} + \gamma_3 x_{ij}^2$$

for all $i = 1, 2$ and $j = 1, 2, 3$, where $\mu, \phi_1, \phi_2, \gamma_1, \gamma_2,$ and γ_3 are unknown parameters. Define $\boldsymbol{\beta} = (\mu, \phi_1, \phi_2, \gamma_1, \gamma_2, \gamma_3)'$ and $\boldsymbol{y} = (y_{11}, y_{12}, y_{13}, y_{21}, y_{22}, y_{23})'$.

(a) Determine the design matrix $\boldsymbol{X}$ so that $E(\boldsymbol{y}) = \boldsymbol{X\beta}$.

(b) Is $\mu + \phi_1$ estimable? Prove that your answer is correct.

(c) $\mathcal{C}(\boldsymbol{X}) = \mathcal{C}(\boldsymbol{W})$, where

$$\boldsymbol{W} = \begin{bmatrix} 1 & 1 & -1 & 0 & 1 \\ 1 & 1 & 0 & 0 & -2 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 & -2 \\ 1 & -1 & 0 & 1 & 1 \end{bmatrix}.$$

Use this fact to find $\hat{\boldsymbol{\beta}}$ such that $\boldsymbol{X}\hat{\boldsymbol{\beta}} = \boldsymbol{P_X y}$.