Best Linear Unbiased Prediction (BLUP)

Best Linear Unbiased Prediction (BLUP):

Suppose

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon},$$

where

$$E(\boldsymbol{\varepsilon}) = \boldsymbol{0}, \quad \text{and} \quad \text{Var}(\boldsymbol{\varepsilon}) = \boldsymbol{\Sigma} = \sigma^2 \boldsymbol{V}.$$

Initially, we will assume the Aitken model holds so that $\sigma^2 > 0$ is unknown and $\boldsymbol{V}$ is a known symmetric and PD matrix.

Suppose u is a random variable with mean 0 and finite variance.

A linear predictor $d + \boldsymbol{a}'\boldsymbol{y}$ of $\boldsymbol{c}'\boldsymbol{\beta} + u$ is unbiased iff

$$E(d + \boldsymbol{a}'\boldsymbol{y}) = E(\boldsymbol{c}'\boldsymbol{\beta} + u) = \boldsymbol{c}'\boldsymbol{\beta} \quad \forall\ \boldsymbol{\beta} \in \mathbb{R}^p.$$

$\boldsymbol{c}'\boldsymbol{\beta} + u$ is predictable iff $\exists$ an unbiased linear predictor of $\boldsymbol{c}'\boldsymbol{\beta} + u$.

Note that $\boldsymbol{c}'\boldsymbol{\beta} + u$ is predictable $\iff \exists\, \boldsymbol{a} \ni \boldsymbol{c}' = \boldsymbol{a}'\boldsymbol{X}$.

This follows from the same argument used to show that $\boldsymbol{c}'\boldsymbol{\beta}$ estimable $\iff \exists\, \boldsymbol{a} \ni \boldsymbol{c}' = \boldsymbol{a}'\boldsymbol{X}$.

Also, $d + \boldsymbol{a}'\boldsymbol{y}$ is an unbiased predictor of $\boldsymbol{c}'\boldsymbol{\beta} + u$ iff

$$d = 0 \quad \text{and} \quad \boldsymbol{c}' = \boldsymbol{a}'\boldsymbol{X},$$

as argued for the case of an unbiased estimator.

Suppose $\hat{w}$ is a predictor of a random variable w.

The mean squared error (MSE) of $\hat{w}$ as a predictor of w is

$$E(\hat{w} - w)^2.$$

Suppose

$$\text{Cov}(\boldsymbol{\varepsilon}, u) = \sigma^2 \boldsymbol{k}$$

for some known vector $\boldsymbol{k}$.

Suppose $\boldsymbol{c}'\boldsymbol{\beta} + u$ is predictable.

Prove that $\boldsymbol{c}'\hat{\boldsymbol{\beta}}_{\text{GLS}} + \hat{u}$ has lowest MSE among all unbiased linear predictors of $\boldsymbol{c}'\boldsymbol{\beta} + u$, where

$$\hat{u} \equiv [\text{Cov}(\boldsymbol{\varepsilon}, u)]'[\text{Var}(\boldsymbol{\varepsilon})]^{-1}(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}}) = \boldsymbol{k}'\boldsymbol{V}^{-1}(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}}).$$

Proof:

First note that

$$\boldsymbol{c}'\hat{\boldsymbol{\beta}}_{\text{GLS}} + \hat{u} = \boldsymbol{c}'\hat{\boldsymbol{\beta}}_{\text{GLS}} + \boldsymbol{k}'\boldsymbol{V}^{-1}(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}})$$

has expectation

$$\boldsymbol{c}'\boldsymbol{\beta} + \boldsymbol{k}'\boldsymbol{V}^{-1}(\boldsymbol{X}\boldsymbol{\beta} - \boldsymbol{X}\boldsymbol{\beta}) = \boldsymbol{c}'\boldsymbol{\beta}.$$

Thus, $\boldsymbol{c}'\hat{\boldsymbol{\beta}}_{\text{GLS}} + \hat{u}$ is an unbiased predictor of $\boldsymbol{c}'\boldsymbol{\beta} + u$.

Now note that

$$\begin{aligned}
\boldsymbol{c}'\hat{\boldsymbol{\beta}}_{\text{GLS}} + \hat{u} &= \boldsymbol{c}'\hat{\boldsymbol{\beta}}_{\text{GLS}} + \boldsymbol{k}'\boldsymbol{V}^{-1}(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}}) \\
&= (\boldsymbol{c}' - \boldsymbol{k}'\boldsymbol{V}^{-1}\boldsymbol{X})\hat{\boldsymbol{\beta}}_{\text{GLS}} + \boldsymbol{k}'\boldsymbol{V}^{-1}\boldsymbol{y} \\
&= (\boldsymbol{c}' - \boldsymbol{k}'\boldsymbol{V}^{-1}\boldsymbol{X})(\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y} + \boldsymbol{k}'\boldsymbol{V}^{-1}\boldsymbol{y} \\
&= [(\boldsymbol{c}' - \boldsymbol{k}'\boldsymbol{V}^{-1}\boldsymbol{X})(\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{V}^{-1} + \boldsymbol{k}'\boldsymbol{V}^{-1}]\boldsymbol{y} \\
&\equiv \boldsymbol{b}'\boldsymbol{y} \quad \text{(a linear estimator.)}
\end{aligned}$$

Because

$$\boldsymbol{c}'\hat{\beta}_{\text{GLS}} + \hat{u} = \boldsymbol{b}'\boldsymbol{y}$$

is an unbiased predictor, we know

$$\boldsymbol{b}'X = \boldsymbol{c}'.$$

Let $\boldsymbol{a}'\boldsymbol{y}$ be any other linear unbiased predictor of $\boldsymbol{c}'\boldsymbol{\beta} + u$. Then $\boldsymbol{a}'\boldsymbol{X} = \boldsymbol{c}'$.

The MSE of $\boldsymbol{a}'\boldsymbol{y}$ is

$$\begin{aligned} E[\boldsymbol{a}'\boldsymbol{y} - (\boldsymbol{c}'\boldsymbol{\beta} + u)]^2 &= \text{Var}[\boldsymbol{a}'\boldsymbol{y} - (\boldsymbol{c}'\boldsymbol{\beta} + u)] \\ &= \text{Var}(\boldsymbol{a}'\boldsymbol{y} - u) \\ &= \text{Var}(\boldsymbol{a}'\boldsymbol{y} - \boldsymbol{b}'\boldsymbol{y} - u + \boldsymbol{b}'\boldsymbol{y}) \\ &= \text{Var}[(\boldsymbol{a} - \boldsymbol{b})'\boldsymbol{y}] + \text{Var}(\boldsymbol{b}'\boldsymbol{y} - u) \\ &\quad +2\text{Cov}[(\boldsymbol{a} - \boldsymbol{b})'\boldsymbol{y}, \boldsymbol{b}'\boldsymbol{y} - u]. \end{aligned}$$

Now

$$
\begin{aligned}
\text{Cov}[(\boldsymbol{a}-\boldsymbol{b})'\boldsymbol{y}, \boldsymbol{b}'\boldsymbol{y}-u] &= (\boldsymbol{a}-\boldsymbol{b})'\text{Var}(\boldsymbol{y})\boldsymbol{b} - \text{Cov}[(\boldsymbol{a}-\boldsymbol{b})'\boldsymbol{y}, u] \\
&= (\boldsymbol{a}-\boldsymbol{b})'\sigma^2\boldsymbol{V}\boldsymbol{b} - (\boldsymbol{a}-\boldsymbol{b})'\text{Cov}(\boldsymbol{y}, u) \\
&= \sigma^2(\boldsymbol{a}-\boldsymbol{b})'\boldsymbol{V}\boldsymbol{b} - (\boldsymbol{a}-\boldsymbol{b})'\text{Cov}(\boldsymbol{\varepsilon}, u) \\
&= \sigma^2(\boldsymbol{a}-\boldsymbol{b})'\boldsymbol{V}\boldsymbol{b} - \sigma^2(\boldsymbol{a}-\boldsymbol{b})'\boldsymbol{k} \\
&= \sigma^2(\boldsymbol{a}-\boldsymbol{b})'(\boldsymbol{V}\boldsymbol{b}-\boldsymbol{k}).
\end{aligned}
$$

Now

$$
\begin{aligned}
\boldsymbol{V}\boldsymbol{b}-\boldsymbol{k} &= \boldsymbol{V}[\boldsymbol{V}^{-1}\boldsymbol{X}[(\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-}]'(\boldsymbol{c}-\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{k})+\boldsymbol{V}^{-1}\boldsymbol{k}]-\boldsymbol{k} \\
&= \boldsymbol{X}[(\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-}]'(\boldsymbol{c}-\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{k}).
\end{aligned}
$$

Thus, the covariance term is

$$
\sigma^2(\boldsymbol{a}-\boldsymbol{b})'\boldsymbol{X}[(\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-}]'(\boldsymbol{c}-\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{k}),
$$

which is 0 because

$$
\boldsymbol{a}'\boldsymbol{X}=\boldsymbol{b}'\boldsymbol{X}=\boldsymbol{c}'.
$$

Thus, we have

$$\begin{aligned}
\text{MSE}(\boldsymbol{a}'\boldsymbol{y}) &= \text{Var}[(\boldsymbol{a}-\boldsymbol{b})'\boldsymbol{y}] + \text{Var}(\boldsymbol{b}'\boldsymbol{y}-u) \\
&= \text{Var}[(\boldsymbol{a}-\boldsymbol{b})'\boldsymbol{y}] + \text{Var}[\boldsymbol{b}'\boldsymbol{y}-(\boldsymbol{c}'\boldsymbol{\beta}+u)] \\
&= \text{Var}[(\boldsymbol{a}-\boldsymbol{b})'\boldsymbol{y}] + E[\boldsymbol{b}'\boldsymbol{y}-(\boldsymbol{c}'\boldsymbol{\beta}+u)]^2 \\
&= \text{Var}[(\boldsymbol{a}-\boldsymbol{b})'\boldsymbol{y}] + \text{MSE}(\boldsymbol{b}'\boldsymbol{y}).
\end{aligned}$$

Thus,

$$\mathrm{MSE}(\boldsymbol{a}'\boldsymbol{y}) \geq \mathrm{MSE}(\boldsymbol{b}'\boldsymbol{y})$$

with equality iff

$$\mathrm{Var}[(\boldsymbol{a}-\boldsymbol{b})'\boldsymbol{y}] = 0 \Longleftrightarrow \boldsymbol{a} = \boldsymbol{b}.$$

Thus $\boldsymbol{c}'\hat{\boldsymbol{\beta}}_{\mathrm{GLS}} + \hat{u}$ is the unique best linear unbiased predictor (BLUP) of $\boldsymbol{c}'\boldsymbol{\beta} + u$. $\square$

In practice,

$$\boldsymbol{\Sigma} = \sigma^2 \boldsymbol{V}$$

involves unknown variance components in addition to the unknown σ^2.

We replace unknown variance components in $\boldsymbol{c}'\hat{\boldsymbol{\beta}}_{\text{GLS}} + \hat{u}$ by the estimates to obtain “empirical” BLUPs (eBLUPs).

This typically results in a nonlinear predictor whose properties are not so well characterized.

Example:

Suppose y_{ij} is the average monthly milk production of the j^{th} daughter of sire i.

Suppose

$$y_{ij} = \mu + s_i + e_{ij} \quad \text{for } i = 1, \ldots, t; \; j = 1, \ldots, n_i,$$

where $s_1, \ldots, s_t$ are iid with mean 0 and variance σ_s^2, independent of the e_{ij} terms, which are iid with mean 0 and variance σ_e^2.

Suppose σ_s^2/σ_e^2 is a known constant d.

Find an expression for the BLUP of $\mu + s_1$.

Let $\boldsymbol{y} = [y_{11}, \ldots, y_{1n_1}, y_{21}, \ldots, y_{2n_2}, \ldots, y_{tn_t}]'$.

Let $\boldsymbol{X} = \underset{n\times 1}{\mathbf{1}}$ where $n = \sum_{i=1}^{t} n_i$.

Let $\boldsymbol{\beta} = [\mu]$.

Let $\boldsymbol{\varepsilon} = \begin{bmatrix} s_1 + e_{11} \\ \vdots \\ s_1 + e_{1n_1} \\ s_2 + e_{21} \\ \vdots \\ s_2 + e_{2n_2} \\ \vdots \\ s_t + e_{tn_t} \end{bmatrix}$.

Then $\mathrm{Var}(\boldsymbol{\varepsilon})$ is block diagonal with blocks of sizes $n_1 \times n_1, n_2 \times n_2, \ldots, n_t \times n_t$.

Each block has $\sigma_s^2 + \sigma_e^2$ on the diagonal and σ_s^2 on the off-diagonal.

The i^{th} block is

$$\sigma_e^2 \underset{n_i \times n_i}{\boldsymbol{I}} + \sigma_s^2 \underset{n_i \times n_i}{\mathbf{1}\mathbf{1}'} = \sigma_e^2[\boldsymbol{I} + d\mathbf{1}\mathbf{1}'].$$

We wish to predict $\boldsymbol{c}'\boldsymbol{\beta} + u$, where $\boldsymbol{c} = [1]$, $\boldsymbol{\beta} = [\mu]$ and $u = s_1$.

$$\begin{aligned} \text{Cov}(\boldsymbol{\varepsilon}, u) &= \text{Cov}(\boldsymbol{\varepsilon}, s_1) = \begin{bmatrix} \sigma_s^2 \underset{n_1 \times 1}{\mathbf{1}} \\ \underset{(n-n_1) \times 1}{\mathbf{0}} \end{bmatrix} \\ &= \sigma_e^2 \begin{bmatrix} d \underset{n_1 \times 1}{\mathbf{1}} \\ \underset{(n-n_1) \times 1}{\mathbf{0}} \end{bmatrix} \equiv \sigma_e^2 \boldsymbol{k}. \end{aligned}$$

We need $\hat{\boldsymbol{\beta}}_{\text{GLS}}$, a solution to the Aitken equations

$$\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X}\boldsymbol{\beta} = \boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y},$$

where

$$\boldsymbol{V} = diag(\underset{n_1 \times n_1}{\boldsymbol{I}} + d\,\underset{n_1 \times n_1}{\mathbf{11}'}, \ldots, \underset{n_t \times n_t}{\boldsymbol{I}} + d\,\underset{n_t \times n_t}{\mathbf{11}'}).$$

$$\left(\underset{n_i \times n_i}{\boldsymbol{I}} + d\,\underset{n_i \times n_i}{\mathbf{11}'}\right)^{-1} = \underset{n_i \times n_i}{\boldsymbol{I}} - \frac{d}{1 + dn_i}\underset{n_i \times n_i}{\mathbf{11}'}.$$

Thus,

$$\begin{aligned}
\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X} &= \mathbf{1}'\boldsymbol{V}^{-1}\mathbf{1} \\
&= \sum_{i=1}^{t} \mathbf{1}'\left[\boldsymbol{I} - \frac{d}{1+dn_i}\mathbf{1}\mathbf{1}'\right]\mathbf{1} \\
&= \sum_{i=1}^{t} \left(n_i - \frac{dn_i^2}{1+dn_i}\right) \\
&= \sum_{i=1}^{t} \frac{n_i}{1+dn_i}.
\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y} &= \sum_{i=1}^{t} \mathbf{1}' \left[\boldsymbol{I} - \frac{d}{1+dn_i} \mathbf{1}\mathbf{1}' \right] \boldsymbol{y}_i \\
&\quad (\text{where } \boldsymbol{y}_i = [y_{i1}, \ldots, y_{in_i}]') \\
&= \sum_{i=1}^{t} \left(y_{i\cdot} - \frac{dn_i}{1+dn_i} y_{i\cdot} \right) \\
&= \sum_{i=1}^{t} \left(1 - \frac{dn_i}{1+dn_i} \right) y_{i\cdot} \\
&= \sum_{i=1}^{t} \frac{n_i}{1+dn_i} \bar{y}_{i\cdot}.
\end{aligned}
$$

Thus,

$$\begin{aligned}
\hat{\boldsymbol{\beta}}_{\text{GLS}} &= \hat{\mu}_{\text{GLS}} = (\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y} \\
&= \frac{\sum_{i=1}^{t} \frac{n_i}{1+dn_i}\bar{y}_{i\cdot}}{\sum_{i=1}^{t} \frac{n_i}{1+dn_i}} \\
&= \frac{\sum_{i=1}^{t}(\sigma_e^2/n_i + \sigma_s^2)^{-1}\bar{y}_{i\cdot}}{\sum_{i=1}^{t}(\sigma_e^2/n_i + \sigma_s^2)^{-1}}.
\end{aligned}$$

Note that each $\bar{y}_{i\cdot}$ is an unbiased estimator of μ with variance

$$\sigma_e^2/n_i + \sigma_s^2.$$

Thus, $\hat{\mu}_{\text{GLS}}$ is a weighted average of independent unbiased estimators of μ, with inverse variances of estimators as weights.

Now,

$$\hat{s}_1 = \hat{u} = \boldsymbol{k}'\boldsymbol{V}^{-1}(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}})$$

$$\begin{aligned}
\boldsymbol{k}'\boldsymbol{V}^{-1}\boldsymbol{y} &= d\mathbf{1}'\left[\boldsymbol{I} - \frac{d}{1+dn_1}\mathbf{1}\mathbf{1}'\right]\boldsymbol{y}_1 \\
&= dy_{1\cdot} - \frac{d^2n_1}{1+dn_1}y_{1\cdot} \\
&= \frac{d + d^2n_1 - d^2n_1}{1+dn_1}y_{1\cdot} \\
&= \frac{dn_1}{1+dn_1}\bar{y}_{1\cdot}.
\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{k}'\boldsymbol{V}^{-1}\boldsymbol{X}\hat{\boldsymbol{\beta}}_{\mathrm{GLS}} &= d\mathbf{1}'\left[\boldsymbol{I} - \frac{d}{1+dn_1}\mathbf{1}\mathbf{1}'\right]\mathbf{1}\hat{\mu}_{\mathrm{GLS}} \\
&= \left(dn_1 - \frac{d^2n_1^2}{1+dn_1}\right)\hat{\mu}_{\mathrm{GLS}} \\
&= \frac{dn_1}{1+dn_1}\hat{\mu}_{\mathrm{GLS}}.
\end{aligned}
$$

Thus, the BLUP of $\mu + s_1$ is

$$
\begin{aligned}
&\hat{\mu}_{\text{GLS}} + \frac{dn_1}{1 + dn_1}\bar{y}_{1\cdot} - \frac{dn_1}{1 + dn_1}\hat{\mu}_{\text{GLS}} \\
&= \frac{dn_1}{1 + dn_1}\bar{y}_{1\cdot} + \left(1 - \frac{dn_1}{1 + dn_1}\right)\hat{\mu}_{\text{GLS}} \\
&= \frac{\sigma_s^2}{\sigma_e^2/n_1 + \sigma_s^2}\bar{y}_{1\cdot} + \frac{\sigma_e^2/n_1}{\sigma_e^2/n_1 + \sigma_s^2}\hat{\mu}_{\text{GLS}}.
\end{aligned}
$$

The mean squared error (MSE) of a predictor $\hat{\boldsymbol{w}}$ of a random vector $\boldsymbol{w}$ is

$$\text{MSE}(\hat{\boldsymbol{w}}) = E[(\hat{\boldsymbol{w}} - \boldsymbol{w})(\hat{\boldsymbol{w}} - \boldsymbol{w})'].$$

Now suppose $\underset{q\times 1}{\boldsymbol{u}}$ is a random vector with mean $\mathbf{0}$ and $\text{Cov}(\boldsymbol{\varepsilon}, \boldsymbol{u}) = \sigma^2 \underset{n\times q}{\boldsymbol{K}}$.

The BLUP of predictable $\boldsymbol{C\beta} + \boldsymbol{u}$ is $\boldsymbol{C}\hat{\boldsymbol{\beta}}_{\text{GLS}} + \hat{\boldsymbol{u}}$, where

$$\hat{\boldsymbol{u}} \equiv [\text{Cov}(\boldsymbol{\varepsilon}, \boldsymbol{u})]'[\text{Var}(\boldsymbol{\varepsilon})]^{-1}(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}}) = \boldsymbol{K}'\boldsymbol{V}^{-1}(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}}).$$

$\boldsymbol{C}\hat{\boldsymbol{\beta}}_{\text{GLS}} + \hat{\boldsymbol{u}}$ is the "best" linear unbiased predictor in the sense that

$$\text{MSE}(\hat{\boldsymbol{w}}) - \text{MSE}(\boldsymbol{C}\hat{\boldsymbol{\beta}}_{\text{GLS}} + \hat{\boldsymbol{u}})$$

is nonnegative definite $\forall\ \hat{\boldsymbol{w}}$, a linear unbiased predictor of $\boldsymbol{C\beta} + \boldsymbol{u}$.

As for the case of univariate prediction, we replace unknown variance components in $\boldsymbol{C}\hat{\boldsymbol{\beta}}_{\text{GLS}} + \hat{\boldsymbol{u}}$ by their estimates to obtain an eBLUP whenever necessary.

For example, suppose

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{Zu} + \boldsymbol{e}, \quad \text{where}$$

$$E(\boldsymbol{u}) = \boldsymbol{0} \qquad E(\boldsymbol{e}) = \boldsymbol{0}$$

$$\text{Var}(\boldsymbol{u}) = \boldsymbol{G} \qquad \text{Var}(\boldsymbol{e}) = \boldsymbol{R}$$

$$\text{Cov}(\boldsymbol{u}, \boldsymbol{e}) = \boldsymbol{0}.$$

Then, with $\boldsymbol{\varepsilon} = \boldsymbol{Z}\boldsymbol{u} + \boldsymbol{e}$, we have

$$\text{Var}(\boldsymbol{\varepsilon}) = \boldsymbol{Z}\boldsymbol{G}\boldsymbol{Z}' + \boldsymbol{R}$$

and

$$\begin{aligned}
\text{Cov}(\boldsymbol{\varepsilon}, \boldsymbol{u}) &= \text{Cov}(\boldsymbol{Z}\boldsymbol{u} + \boldsymbol{e}, \boldsymbol{u}) \\
&= \text{Cov}(\boldsymbol{Z}\boldsymbol{u}, \boldsymbol{u}) \\
&= \boldsymbol{Z}\text{Cov}(\boldsymbol{u}, \boldsymbol{u}) \\
&= \boldsymbol{Z}\text{Var}(\boldsymbol{u}) \\
&= \boldsymbol{Z}\boldsymbol{G}.
\end{aligned}$$

It follows that the BLUP of $\boldsymbol{u}$ is

$$\boldsymbol{G}\boldsymbol{Z}'(\boldsymbol{Z}\boldsymbol{G}\boldsymbol{Z}' + \boldsymbol{R})^{-1}(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}}).$$

Similarly, if $\boldsymbol{D}$ is a known $m \times q$ matrix, the BLUP of $\boldsymbol{Du}$ is

$$\boldsymbol{DGZ}'(\boldsymbol{ZGZ}' + \boldsymbol{R})^{-1}(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}}).$$

When $\boldsymbol{G}$ and $\boldsymbol{R}$ are unknown, we use the eBLUP

$$\boldsymbol{D}\hat{\boldsymbol{G}}\boldsymbol{Z}'(\boldsymbol{Z}\hat{\boldsymbol{G}}\boldsymbol{Z}' + \hat{\boldsymbol{R}})^{-1}(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}}).$$

Note that computing $\hat{\boldsymbol{\beta}}_{\text{GLS}}$ or computing the BLUP of $\boldsymbol{Du}$ requires inversion of the $n \times n$ matrix $\boldsymbol{ZGZ}' + \boldsymbol{R}$.

This can be computationally expensive.

We often assume that

$$\boldsymbol{R} = \sigma_e^2 \boldsymbol{I},$$

but even then

$$\boldsymbol{ZGZ}' + \sigma_e^2 \boldsymbol{I}$$

is a $n \times n$ matrix that may be difficult to invert.

Until further notice, assume

$$\boldsymbol{R} = \sigma_e^2 \boldsymbol{I}$$

for some unknown $\sigma_e^2 > 0$.

Define $\boldsymbol{H} = \frac{1}{\sigma_e^2}\boldsymbol{G}$. Then

$$\begin{aligned} \text{Var}(\boldsymbol{y}) &= \boldsymbol{ZGZ}' + \sigma_e^2 \boldsymbol{I} \\ &= \sigma_e^2(\boldsymbol{ZHZ}' + \boldsymbol{I}) = \sigma_e^2 \boldsymbol{V} \\ &\quad \text{where } \boldsymbol{V} = \boldsymbol{ZHZ}' + \boldsymbol{I}. \end{aligned}$$

C.R. Henderson introduced the Mixed Model Equations(MME)

$$\begin{bmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z} \\ \boldsymbol{Z}'\boldsymbol{X} & \boldsymbol{H}^{-1} + \boldsymbol{Z}'\boldsymbol{Z} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{\beta}} \\ \tilde{\boldsymbol{u}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}'\boldsymbol{y} \\ \boldsymbol{Z}'\boldsymbol{y} \end{bmatrix}.$$

Recall that the Aitken equations are

$$\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X}\tilde{\boldsymbol{\beta}} = \boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y},$$

which in our special case are equivalent to

$$\boldsymbol{X}'(\boldsymbol{ZHZ}' + \boldsymbol{I})^{-1}\boldsymbol{X}\tilde{\boldsymbol{\beta}} = \boldsymbol{X}'(\boldsymbol{ZHZ}' + \boldsymbol{I})^{-1}\boldsymbol{y}.$$

Henderson showed that

(1) The MME are consistent.

(2) If $\begin{bmatrix} \tilde{\boldsymbol{\beta}} \\ \tilde{\boldsymbol{u}} \end{bmatrix}$ solve the MME, then $\tilde{\boldsymbol{\beta}}$ is a solution to the AE (equivalently, $\boldsymbol{X}\tilde{\boldsymbol{\beta}} = \boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}}$) and $\tilde{\boldsymbol{u}}$ is the BLUP of $\boldsymbol{u}$.

(3) Conversely, if $\tilde{\boldsymbol{\beta}}$ solves the AE, then the MME have a solution whose leading subvector is $\tilde{\boldsymbol{\beta}}$.

A nice thing about this result is that we can find $\hat{\boldsymbol{\beta}}_{\text{GLS}}$ and the BLUP of $\boldsymbol{u}$ (or $\boldsymbol{Du}$) without inverting the $n \times n$ matrix $(\boldsymbol{ZGZ}' + \sigma_e^2 \boldsymbol{I})$.

$$\begin{bmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z} \\ \boldsymbol{Z}'\boldsymbol{X} & \boldsymbol{H}^{-1} + \boldsymbol{Z}'\boldsymbol{Z} \end{bmatrix}$$

is $(p+q) \times (p+q)$. In some problems, $p + q \ll n$.

By (2), the BLUP of a predictable $\boldsymbol{C\beta} + \boldsymbol{Du}$ is given by

$$\boldsymbol{C}\tilde{\boldsymbol{\beta}} + \boldsymbol{D}\tilde{\boldsymbol{u}}$$

where $\begin{bmatrix} \tilde{\boldsymbol{\beta}} \\ \tilde{\boldsymbol{u}} \end{bmatrix}$ solves the MME.

To prove Henderson's results, we begin with the following lemma.

Lemma H1: Suppose $\boldsymbol{A}$ is a symmetric and positive definite matrix. Suppose $\boldsymbol{W}$ is any matrix with number of columns equal to number of rows of $\boldsymbol{A}$. Then

$$\boldsymbol{A}\boldsymbol{W}'(\boldsymbol{I} + \boldsymbol{W}\boldsymbol{A}\boldsymbol{W}')^{-1} = (\boldsymbol{A}^{-1} + \boldsymbol{W}'\boldsymbol{W})^{-1}\boldsymbol{W}'.$$

Proof:

First show that

$$\boldsymbol{I} + \boldsymbol{W}\boldsymbol{A}\boldsymbol{W}' \quad \text{and} \quad \boldsymbol{A}^{-1} + \boldsymbol{W}'\boldsymbol{W}$$

are each nonsingular.

$$\begin{aligned}\forall\, \boldsymbol{x} \neq \boldsymbol{0},\ \boldsymbol{x}'(\boldsymbol{I} + \boldsymbol{WAW}')\boldsymbol{x} &= \boldsymbol{x}'\boldsymbol{x} + \boldsymbol{x}'\boldsymbol{WAW}'\boldsymbol{x} \\ &= ||\boldsymbol{x}||^2 + ||\boldsymbol{A}^{1/2}\boldsymbol{W}'\boldsymbol{x}||^2 \\ &\geq ||\boldsymbol{x}||^2 > 0.\end{aligned}$$

Thus, $\boldsymbol{I} + \boldsymbol{WAW}'$ is PD and $\therefore$ nonsingular.

$$\begin{aligned}\forall\, \boldsymbol{x} \neq \boldsymbol{0},\ \boldsymbol{x}'(\boldsymbol{A}^{-1} + \boldsymbol{W}'\boldsymbol{W})\boldsymbol{x} &= \boldsymbol{x}'\boldsymbol{A}^{-1}\boldsymbol{x} + \boldsymbol{x}'\boldsymbol{W}'\boldsymbol{W}\boldsymbol{x} \\ &= \boldsymbol{x}'\boldsymbol{A}^{-1}\boldsymbol{x} + ||\boldsymbol{W}\boldsymbol{x}||^2 \\ &\geq \boldsymbol{x}'\boldsymbol{A}^{-1}\boldsymbol{x} > 0\end{aligned}$$

$\because \boldsymbol{A}$ PD $\Longrightarrow \boldsymbol{A}^{-1}$ PD. $\therefore \boldsymbol{A}^{-1} + \boldsymbol{W}'\boldsymbol{W}$ is nonsingular.

Now show that

$$(\boldsymbol{A}^{-1} + \boldsymbol{W}'\boldsymbol{W})\boldsymbol{A}\boldsymbol{W}' = \boldsymbol{W}'(\boldsymbol{I} + \boldsymbol{W}\boldsymbol{A}\boldsymbol{W}')$$

and then complete the proof.

$$
\begin{aligned}
(\boldsymbol{A}^{-1} + \boldsymbol{W}'\boldsymbol{W})\boldsymbol{A}\boldsymbol{W}' &= \boldsymbol{A}^{-1}\boldsymbol{A}\boldsymbol{W}' + \boldsymbol{W}'\boldsymbol{W}\boldsymbol{A}\boldsymbol{W}' \\
&= \boldsymbol{W}' + \boldsymbol{W}'\boldsymbol{W}\boldsymbol{A}\boldsymbol{W}' \\
&= \boldsymbol{W}'(\boldsymbol{I} + \boldsymbol{W}\boldsymbol{A}\boldsymbol{W}').
\end{aligned}
$$

Multiplying on the left by

$$(\boldsymbol{A}^{-1} + \boldsymbol{W}'\boldsymbol{W})^{-1}$$

gives

$$\boldsymbol{A}\boldsymbol{W}' = (\boldsymbol{A}^{-1} + \boldsymbol{W}'\boldsymbol{W})^{-1}\boldsymbol{W}'(\boldsymbol{I} + \boldsymbol{W}\boldsymbol{A}\boldsymbol{W}').$$

Multiplying on the right by $(\boldsymbol{I} + \boldsymbol{W}\boldsymbol{A}\boldsymbol{W}')^{-1}$ gives the result. $\square$

Now use Lemma H1 to prove Lemma H2:

$$\begin{aligned} \boldsymbol{V}^{-1} &= (\boldsymbol{I} + \boldsymbol{Z}\boldsymbol{H}\boldsymbol{Z}')^{-1} \\ &= \boldsymbol{I} - \boldsymbol{Z}(\boldsymbol{H}^{-1} + \boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'. \end{aligned}$$

Proof:

Applying the Lemma H1 with $\boldsymbol{H} = \boldsymbol{A}$ and $\boldsymbol{Z} = \boldsymbol{W}$ yields

$$(\boldsymbol{H}^{-1} + \boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}' = \boldsymbol{H}\boldsymbol{Z}'(\boldsymbol{I} + \boldsymbol{Z}\boldsymbol{H}\boldsymbol{Z}')^{-1}. \qquad (*)$$

Thus,

$$\begin{aligned}
&[\boldsymbol{I} - \boldsymbol{Z}(\boldsymbol{H}^{-1} + \boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'](\boldsymbol{I} + \boldsymbol{Z}\boldsymbol{H}\boldsymbol{Z}') \\
&= [\boldsymbol{I} - \boldsymbol{Z}\boldsymbol{H}\boldsymbol{Z}'(\boldsymbol{I} + \boldsymbol{Z}\boldsymbol{H}\boldsymbol{Z}')^{-1}](\boldsymbol{I} + \boldsymbol{Z}\boldsymbol{H}\boldsymbol{Z}') \text{ by } (*) \\
&= \boldsymbol{I} + \boldsymbol{Z}\boldsymbol{H}\boldsymbol{Z}' - \boldsymbol{Z}\boldsymbol{H}\boldsymbol{Z}' = \boldsymbol{I}.
\end{aligned}$$

□

Now let's prove Henderson's results beginning with (2).

Suppose $\begin{bmatrix} \tilde{\boldsymbol{\beta}} \\ \tilde{\boldsymbol{u}} \end{bmatrix}$ solves the MME.

Then

$$\boldsymbol{X}'\boldsymbol{X}\tilde{\boldsymbol{\beta}} + \boldsymbol{X}'\boldsymbol{Z}\tilde{\boldsymbol{u}} = \boldsymbol{X}'\boldsymbol{y}$$

and

$$\boldsymbol{Z}'\boldsymbol{X}\tilde{\boldsymbol{\beta}} + (\boldsymbol{H}^{-1} + \boldsymbol{Z}'\boldsymbol{Z})\tilde{\boldsymbol{u}} = \boldsymbol{Z}'\boldsymbol{y}.$$

The 2^{nd} equation

$$\Longrightarrow (\boldsymbol{H}^{-1} + \boldsymbol{Z}'\boldsymbol{Z})\tilde{\boldsymbol{u}} = \boldsymbol{Z}'\boldsymbol{y} - \boldsymbol{Z}'\boldsymbol{X}\tilde{\boldsymbol{\beta}}$$

$$\Longrightarrow \tilde{\boldsymbol{u}} = (\boldsymbol{H}^{-1} + \boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'(\boldsymbol{y} - \boldsymbol{X}\tilde{\boldsymbol{\beta}})$$

$$\Longrightarrow \tilde{\boldsymbol{u}} = \boldsymbol{H}\boldsymbol{Z}'(\boldsymbol{I} + \boldsymbol{Z}\boldsymbol{H}\boldsymbol{Z}')^{-1}(\boldsymbol{y} - \boldsymbol{X}\tilde{\boldsymbol{\beta}}) \text{ by } (*)$$

$$\Longrightarrow \tilde{\boldsymbol{u}} = \boldsymbol{H}\boldsymbol{Z}'\boldsymbol{V}^{-1}(\boldsymbol{y} - \boldsymbol{X}\tilde{\boldsymbol{\beta}}) = \boldsymbol{G}\boldsymbol{Z}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{y} - \boldsymbol{X}\tilde{\boldsymbol{\beta}}).$$

Now this last expression for $\tilde{\boldsymbol{u}}$ indicates that $\tilde{\boldsymbol{u}}$ is the BLUP for $\boldsymbol{u}$ as long as $\tilde{\boldsymbol{\beta}}$ solves the Aitken equations.

To show this, we examine the 1^{st} MME.

The 1^{st} MME is

$$\boldsymbol{X}'\boldsymbol{X}\tilde{\boldsymbol{\beta}} + \boldsymbol{X}'\boldsymbol{Z}\tilde{\boldsymbol{u}} = \boldsymbol{X}'\boldsymbol{y}.$$

We have shown

$$\tilde{\boldsymbol{u}} = (\boldsymbol{H}^{-1} + \boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'(\boldsymbol{y} - \boldsymbol{X}\tilde{\boldsymbol{\beta}}).$$

Combining these equations gives

$$\boldsymbol{X}'\boldsymbol{X}\tilde{\boldsymbol{\beta}} + \boldsymbol{X}'\boldsymbol{Z}(\boldsymbol{H}^{-1} + \boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'(\boldsymbol{y} - \boldsymbol{X}\tilde{\boldsymbol{\beta}}) = \boldsymbol{X}'\boldsymbol{y}.$$

By Lemma H2,

$$\boldsymbol{Z}(\boldsymbol{H}^{-1} + \boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}' = \boldsymbol{I} - \boldsymbol{V}^{-1}.$$

Thus,

$$\begin{aligned}
&\boldsymbol{X}'\boldsymbol{X}\tilde{\boldsymbol{\beta}} + \boldsymbol{X}'(\boldsymbol{I} - \boldsymbol{V}^{-1})(\boldsymbol{y} - \boldsymbol{X}\tilde{\boldsymbol{\beta}}) = \boldsymbol{X}'\boldsymbol{y} \\
\iff &\boldsymbol{X}'\boldsymbol{X}\tilde{\boldsymbol{\beta}} + \boldsymbol{X}'\boldsymbol{y} - \boldsymbol{X}'\boldsymbol{X}\tilde{\boldsymbol{\beta}} - \boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y} + \boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X}\tilde{\boldsymbol{\beta}} = \boldsymbol{X}'\boldsymbol{y} \\
\iff &\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X}\tilde{\boldsymbol{\beta}} = \boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y}.
\end{aligned}$$

$\therefore$ $\tilde{\boldsymbol{\beta}}$ solves the AE and Henderson's result (2) follows.

Now let's prove (3).

Suppose $\tilde{\beta}$ solves the AE.

Let

$$\begin{aligned}
\tilde{\boldsymbol{u}} &= \boldsymbol{H}\boldsymbol{Z}'\boldsymbol{V}^{-1}(\boldsymbol{y}-\boldsymbol{X}\tilde{\boldsymbol{\beta}}) \\
&= \boldsymbol{H}\boldsymbol{Z}'(\boldsymbol{I}+\boldsymbol{Z}\boldsymbol{H}\boldsymbol{Z}')^{-1}(\boldsymbol{y}-\boldsymbol{X}\tilde{\boldsymbol{\beta}}) \\
&= (\boldsymbol{H}^{-1}+\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'(\boldsymbol{y}-\boldsymbol{X}\tilde{\boldsymbol{\beta}}) \text{ by } (*)
\end{aligned}$$

$$\Longrightarrow (\boldsymbol{H}^{-1}+\boldsymbol{Z}'\boldsymbol{Z})\tilde{\boldsymbol{u}} = \boldsymbol{Z}'(\boldsymbol{y}-\boldsymbol{X}\tilde{\boldsymbol{\beta}}) = \boldsymbol{Z}'\boldsymbol{y}-\boldsymbol{Z}'\boldsymbol{X}\tilde{\boldsymbol{\beta}}$$
$$\Longrightarrow \boldsymbol{Z}'\boldsymbol{X}\tilde{\boldsymbol{\beta}} + (\boldsymbol{H}^{-1}+\boldsymbol{Z}'\boldsymbol{Z})\tilde{\boldsymbol{u}} = \boldsymbol{Z}'\boldsymbol{y}.$$

$\therefore 2^{nd}$ MME is satisfied.

Now

$$\begin{aligned}&\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X}\tilde{\boldsymbol{\beta}} = \boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y}\\ &\Longrightarrow \boldsymbol{X}'[\boldsymbol{I} - \boldsymbol{Z}(\boldsymbol{H}^{-1} + \boldsymbol{Z}'\boldsymbol{Z})\boldsymbol{Z}']\boldsymbol{X}\tilde{\boldsymbol{\beta}} = \boldsymbol{X}'[\boldsymbol{I} - \boldsymbol{Z}(\boldsymbol{H}^{-1} + \boldsymbol{Z}'\boldsymbol{Z})\boldsymbol{Z}']\boldsymbol{y}\end{aligned}$$

(By Lemma H2)

It follows that

$$\boldsymbol{X}'\boldsymbol{X}\tilde{\boldsymbol{\beta}} + \boldsymbol{X}'\boldsymbol{Z}(\boldsymbol{H}^{-1} + \boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'(\boldsymbol{y} - \boldsymbol{X}\tilde{\boldsymbol{\beta}}) = \boldsymbol{X}'\boldsymbol{y}.$$

Now

$$\begin{aligned} \tilde{\boldsymbol{u}} &= (\boldsymbol{H}^{-1} + \boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'(\boldsymbol{y} - \boldsymbol{X}\tilde{\boldsymbol{\beta}}) \\ &\Longrightarrow \boldsymbol{X}'\boldsymbol{X}\tilde{\boldsymbol{\beta}} + \boldsymbol{X}'\boldsymbol{Z}\tilde{\boldsymbol{u}} = \boldsymbol{X}'\boldsymbol{y}. \end{aligned}$$

$\therefore$ The 1^{st} MME is satisfied and result (3) follows.

Result (1) is now easy to prove as follows.

The AE are consistent.

Thus, by Henderson's result (3), the MME are consistent. □

Now suppose $\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{Zu} + \boldsymbol{e}$, where all is as before except that, instead of $\mathrm{Var}(\boldsymbol{e}) = \sigma_e^2\boldsymbol{I}$, $\mathrm{Var}(\boldsymbol{e}) = \boldsymbol{R}$, a symmetric, positive definite variance matrix.

Let $\boldsymbol{S} = \frac{1}{\sigma_e^2}\boldsymbol{R}$, where σ_e^2 is a positive variance parameter.

Consider the transformation

$$\boldsymbol{S}^{-1/2}\boldsymbol{y} = \boldsymbol{S}^{-1/2}\boldsymbol{X\beta} + \boldsymbol{S}^{-1/2}\boldsymbol{Zu} + \boldsymbol{S}^{-1/2}\boldsymbol{e}.$$

Then $\boldsymbol{S}^{-1/2}\boldsymbol{e} \sim N(\mathbf{0}, \sigma_e^2\boldsymbol{I})$, and our previous results with

$$\boldsymbol{S}^{-1/2}\boldsymbol{y}, \boldsymbol{S}^{-1/2}\boldsymbol{X}, \text{ and } \boldsymbol{S}^{-1/2}\boldsymbol{Z} \text{ in place of } \boldsymbol{y}, \boldsymbol{X}, \text{ and } \boldsymbol{Z}$$

yield the mixed model equations

$$\begin{bmatrix} \boldsymbol{X}'\boldsymbol{S}^{-1}\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{S}^{-1}\boldsymbol{Z} \\ \boldsymbol{Z}'\boldsymbol{S}^{-1}\boldsymbol{X} & \boldsymbol{H}^{-1} + \boldsymbol{Z}'\boldsymbol{S}^{-1}\boldsymbol{Z} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{\beta}} \\ \tilde{\boldsymbol{u}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}'\boldsymbol{S}^{-1}\boldsymbol{y} \\ \boldsymbol{Z}'\boldsymbol{S}^{-1}\boldsymbol{y} \end{bmatrix}.$$

Multiplying both sides by $1/\sigma_e^2$ gives

$$\begin{bmatrix} \boldsymbol{X}'\boldsymbol{R}^{-1}\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{R}^{-1}\boldsymbol{Z} \\ \boldsymbol{Z}'\boldsymbol{R}^{-1}\boldsymbol{X} & \boldsymbol{G}^{-1} + \boldsymbol{Z}'\boldsymbol{R}^{-1}\boldsymbol{Z} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{\beta}} \\ \tilde{\boldsymbol{u}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}'\boldsymbol{R}^{-1}\boldsymbol{y} \\ \boldsymbol{Z}'\boldsymbol{R}^{-1}\boldsymbol{y} \end{bmatrix}.$$

This is the more general form of Henderson's Mixed Model Equations.

Finally, let $\hat{\boldsymbol{G}}$ and $\hat{\boldsymbol{R}}$ denote $\boldsymbol{G}$ and $\boldsymbol{R}$ with unknown parameters replaced by their REML estimates.

In practice, we solve

$$\begin{bmatrix} \boldsymbol{X}'\hat{\boldsymbol{R}}^{-1}\boldsymbol{X} & \boldsymbol{X}'\hat{\boldsymbol{R}}^{-1}\boldsymbol{Z} \\ \boldsymbol{Z}'\hat{\boldsymbol{R}}^{-1}\boldsymbol{X} & \hat{\boldsymbol{G}}^{-1} + \boldsymbol{Z}'\hat{\boldsymbol{R}}^{-1}\boldsymbol{Z} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{\beta}} \\ \tilde{\boldsymbol{u}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}'\hat{\boldsymbol{R}}^{-1}\boldsymbol{y} \\ \boldsymbol{Z}'\hat{\boldsymbol{R}}^{-1}\boldsymbol{y} \end{bmatrix},$$

to get solutions needed for computation of BLUEs and BLUPs.