

Tukey's Method

These slides cover Tukey's method for all pairwise comparisons of means in a balanced single-factor ANOVA.

Suppose $Z_1, \dots, Z_k \stackrel{i.i.d.}{\sim} N(0, 1)$.

Suppose $U \sim \chi_m^2$ and U is independent of Z_1, \dots, Z_k .

Let

$$R = \frac{\max_{i=1, \dots, k} Z_i - \min_{i=1, \dots, k} Z_i}{\sqrt{U/m}}.$$

The distribution of R is known as the distribution of the studentized range.

Let $R_{k,m,\alpha}$ denote the upper α quantile of the distribution of the studentized range so that

$$\mathbb{P}(R \leq R_{k,m,\alpha}) = 1 - \alpha.$$

Now suppose

$$y_{ij} = \mu_i + \varepsilon_{ij},$$

where the ε_{ij} terms are iid $N(0, \sigma^2)$ and $i = 1, \dots, t$ and $j = 1, \dots, n$.

Prove that

$$\begin{aligned} \mathbb{P} \left[(\bar{y}_{i\cdot} - \bar{y}_{i^*\cdot}) - \frac{\hat{\sigma}}{\sqrt{n}} R_{t,t(n-1),\alpha} \leq \mu_i - \mu_{i^*} \right. \\ \left. \leq (\bar{y}_{i\cdot} - \bar{y}_{i^*\cdot}) + \frac{\hat{\sigma}}{\sqrt{n}} R_{t,t(n-1),\alpha} \quad \forall i \neq i^* \right] = 1 - \alpha. \end{aligned}$$

The intervals

$$(\bar{y}_{i\cdot} - \bar{y}_{i^*\cdot}) \pm \frac{\hat{\sigma}}{\sqrt{n}} R_{t,t(n-1),\alpha} \quad i \neq i^*$$

are Tukey's $100(1 - \alpha)\%$ simultaneous confidence intervals for all possible pairwise differences $\mu_i - \mu_{i^*}$, $i \neq i^*$.

The previous result shows that these intervals have simultaneous coverage $1 - \alpha$.

For the unbalanced case

$$y_{ij} = \mu_i + \varepsilon_{ij}, \quad i = 1, \dots, t; j = 1, \dots, n_i,$$

the Tukey-Kramer intervals

$$(\bar{y}_{i\cdot} - \bar{y}_{i^*\cdot}) \pm \hat{\sigma} \sqrt{\frac{1/n_i + 1/n_{i^*}}{2}} R_{t,t(n-1),\alpha} \quad i \neq i^*$$

have simultaneous coverage at least $1 - \alpha$.

(See Hayter, A.J. (1984). *Annals of Statistics*.12, 61-75.)