The Cauchy-Schwarz Inequality and Generalizations

Cauchy-Schwarz Inequality:

$$(\boldsymbol{a}'\boldsymbol{b})^2 \leq (\boldsymbol{a}'\boldsymbol{a})(\boldsymbol{b}'\boldsymbol{b})$$

with equality iff $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly dependent.

Proof:

First show that

$$(\boldsymbol{a}'\boldsymbol{b})^2 = (\boldsymbol{a}'\boldsymbol{a})(\boldsymbol{b}'\boldsymbol{b})$$

if $\boldsymbol{a}$ and $\boldsymbol{b}$ are LD.

Now suppose $\boldsymbol{a}$ and $\boldsymbol{b}$ are LI and prove

$$(\boldsymbol{a}'\boldsymbol{b})^2 < (\boldsymbol{a}'\boldsymbol{a})(\boldsymbol{b}'\boldsymbol{b}).$$

Prove the following:

If $\boldsymbol{A}$ is a positive definite symmetric matrix, then

(i) $(\boldsymbol{a}'\boldsymbol{A}\boldsymbol{b})^2 \leq (\boldsymbol{a}'\boldsymbol{A}\boldsymbol{a})(\boldsymbol{b}'\boldsymbol{A}\boldsymbol{b})$ with equality iff $\boldsymbol{a}$ and $\boldsymbol{b}$ LD.

(ii) $(\boldsymbol{a}'\boldsymbol{b})^2 \leq (\boldsymbol{a}'\boldsymbol{A}\boldsymbol{a})(\boldsymbol{b}'\boldsymbol{A}^{-1}\boldsymbol{b})$ with equality iff $\boldsymbol{a}$ and $\boldsymbol{A}^{-1}\boldsymbol{b}$ LD.

Suppose $\underset{p \times p}{\boldsymbol{A}}$ is symmetric and positive definite.

Let $\boldsymbol{b}$ be any nonzero vector in $\mathbb{R}^p$.

Prove that

$$\max_{\boldsymbol{a} \neq \boldsymbol{0}} \frac{(\boldsymbol{a}'\boldsymbol{b})^2}{\boldsymbol{a}'\boldsymbol{A}\boldsymbol{a}} = \boldsymbol{b}'\boldsymbol{A}^{-1}\boldsymbol{b}$$

and that

$$\frac{(\boldsymbol{a}'\boldsymbol{b})^2}{\boldsymbol{a}'\boldsymbol{A}\boldsymbol{a}} = \boldsymbol{b}'\boldsymbol{A}^{-1}\boldsymbol{b}$$

whenever $\boldsymbol{a} = c\boldsymbol{A}^{-1}\boldsymbol{b}$ for $c \in \mathbb{R} \setminus \{0\}$.