The Cauchy-Schwarz Inequality and Generalizations

Cauchy-Schwarz Inequality:

$$(\boldsymbol{a}'\boldsymbol{b})^2 \leq (\boldsymbol{a}'\boldsymbol{a})(\boldsymbol{b}'\boldsymbol{b})$$

with equality iff $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly dependent.

Proof:

First show that

$$(\boldsymbol{a}'\boldsymbol{b})^2 = (\boldsymbol{a}'\boldsymbol{a})(\boldsymbol{b}'\boldsymbol{b})$$

if $\boldsymbol{a}$ and $\boldsymbol{b}$ are LD.

If $\boldsymbol{a} = \mathbf{0}$, the result holds trivially.

If $\boldsymbol{a} \neq \mathbf{0}$ and $\boldsymbol{a}$ and $\boldsymbol{b}$ are LD, then $\exists\ c \in \mathbb{R} \ni$

$$\boldsymbol{b} = c\boldsymbol{a}.$$

$$\begin{aligned} \therefore (\boldsymbol{a}'\boldsymbol{b})^2 &= (\boldsymbol{a}'c\boldsymbol{a})^2 = c^2(\boldsymbol{a}'\boldsymbol{a})^2 \\ &= c^2(\boldsymbol{a}'\boldsymbol{a})(\boldsymbol{a}'\boldsymbol{a}) \\ &= (\boldsymbol{a}'\boldsymbol{a})((c\boldsymbol{a})'(c\boldsymbol{a})) \\ &= (\boldsymbol{a}'\boldsymbol{a})(\boldsymbol{b}'\boldsymbol{b}). \end{aligned}$$

Now suppose $\boldsymbol{a}$ and $\boldsymbol{b}$ are LI and prove

$$(\boldsymbol{a}'\boldsymbol{b})^2 < (\boldsymbol{a}'\boldsymbol{a})(\boldsymbol{b}'\boldsymbol{b}).$$

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are LI, then

$$\|\boldsymbol{a} - c\boldsymbol{b}\|^2 > 0 \quad \forall\, c \in \mathbb{R}.$$

Note that

$$\begin{aligned} \|\boldsymbol{a} - c\boldsymbol{b}\|^2 &= \|\boldsymbol{a}\|^2 + c^2\|\boldsymbol{b}\|^2 - 2c\boldsymbol{a}'\boldsymbol{b} \\ &= \|\boldsymbol{a}\|^2 + \left(c\|\boldsymbol{b}\| - \frac{\boldsymbol{a}'\boldsymbol{b}}{\|\boldsymbol{b}\|}\right)^2 - \left(\frac{\boldsymbol{a}'\boldsymbol{b}}{\|\boldsymbol{b}\|}\right)^2. \end{aligned}$$

Thus, we have

$$\|\boldsymbol{a}\|^2 + \left(c\|\boldsymbol{b}\| - \frac{\boldsymbol{a}'\boldsymbol{b}}{\|\boldsymbol{b}\|}\right)^2 - \left(\frac{\boldsymbol{a}'\boldsymbol{b}}{\|\boldsymbol{b}\|}\right)^2 > 0 \quad \forall\, c \in \mathbb{R}.$$

For $c = \frac{\boldsymbol{a}'\boldsymbol{b}}{\|\boldsymbol{b}\|^2}$, we have

$$\begin{aligned} &\|\boldsymbol{a}\|^2 - \left(\frac{\boldsymbol{a}'\boldsymbol{b}}{\|\boldsymbol{b}\|}\right)^2 > 0 \\ &\Longrightarrow (\boldsymbol{a}'\boldsymbol{b})^2 < \|\boldsymbol{a}\|^2\|\boldsymbol{b}\|^2 = (\boldsymbol{a}'\boldsymbol{a})(\boldsymbol{b}'\boldsymbol{b}). \end{aligned}$$

□

Prove the following:

If $\boldsymbol{A}$ is a positive definite symmetric matrix, then

(i) $(\boldsymbol{a}'\boldsymbol{A}\boldsymbol{b})^2 \leq (\boldsymbol{a}'\boldsymbol{A}\boldsymbol{a})(\boldsymbol{b}'\boldsymbol{A}\boldsymbol{b})$ with equality iff $\boldsymbol{a}$ and $\boldsymbol{b}$ LD.

(ii) $(\boldsymbol{a}'\boldsymbol{b})^2 \leq (\boldsymbol{a}'\boldsymbol{A}\boldsymbol{a})(\boldsymbol{b}'\boldsymbol{A}^{-1}\boldsymbol{b})$ with equality iff $\boldsymbol{a}$ and $\boldsymbol{A}^{-1}\boldsymbol{b}$ LD.

Proof of (i):

Let $\boldsymbol{u} = \boldsymbol{A}^{1/2}\boldsymbol{a}$ and $\boldsymbol{v} = \boldsymbol{A}^{1/2}\boldsymbol{b}$.

By C-S inequality,

$$(\boldsymbol{u}'\boldsymbol{v})^2 \leq (\boldsymbol{u}'\boldsymbol{u})(\boldsymbol{v}'\boldsymbol{v})$$

with equality iff $\boldsymbol{u}$ and $\boldsymbol{v}$ LD.

Now note that

$$\boldsymbol{u}'\boldsymbol{v} = \boldsymbol{a}'\boldsymbol{A}\boldsymbol{b}, \quad \boldsymbol{u}'\boldsymbol{u} = \boldsymbol{a}'\boldsymbol{A}\boldsymbol{a}, \quad \text{and} \quad \boldsymbol{v}'\boldsymbol{v} = \boldsymbol{b}'\boldsymbol{A}\boldsymbol{b}.$$

Thus, we have

$$(\boldsymbol{a}'\boldsymbol{A}\boldsymbol{b})^2 \leq (\boldsymbol{a}'\boldsymbol{A}\boldsymbol{a})(\boldsymbol{b}'\boldsymbol{A}\boldsymbol{b}).$$

Finally, note that

$$c_1\boldsymbol{a} + c_2\boldsymbol{b} = \boldsymbol{0} \iff c_1\boldsymbol{A}^{1/2}\boldsymbol{a} + c_2\boldsymbol{A}^{1/2}\boldsymbol{b} = \boldsymbol{0} \iff c_1\boldsymbol{u} + c_2\boldsymbol{v} = \boldsymbol{0}$$

because $\boldsymbol{A}^{1/2}$ is nonsingular.

Therefore, $\boldsymbol{a},\ \boldsymbol{b}$ LD $\iff \boldsymbol{u},\ \boldsymbol{v}$ LD. $\square$

Proof of (ii):

Let $\boldsymbol{u} = \boldsymbol{A}^{1/2}\boldsymbol{a}$ and $\boldsymbol{v} = \boldsymbol{A}^{-1/2}\boldsymbol{b}$.

By C-S, $(\boldsymbol{u}\boldsymbol{v})^2 \leq (\boldsymbol{u}'\boldsymbol{u})(\boldsymbol{v}'\boldsymbol{v})$ with equality iff $\boldsymbol{u},\ \boldsymbol{v}$ LD.

Now note that $\boldsymbol{u}'\boldsymbol{v} = \boldsymbol{a}'\boldsymbol{b}$, $\boldsymbol{u}'\boldsymbol{u} = \boldsymbol{a}'\boldsymbol{A}\boldsymbol{a}$, $\boldsymbol{v}'\boldsymbol{v} = \boldsymbol{b}'\boldsymbol{A}^{-1}\boldsymbol{b}$, and

$$\boldsymbol{u},\ \boldsymbol{v} \text{ LD} \iff \boldsymbol{A}^{1/2}\boldsymbol{a},\ \boldsymbol{A}^{-1/2}\boldsymbol{b} \text{ LD} \iff \boldsymbol{a},\ \boldsymbol{A}^{-1}\boldsymbol{b} \text{ LD}.$$

□

Suppose $\underset{p \times p}{\boldsymbol{A}}$ is symmetric and positive definite.

Let $\boldsymbol{b}$ be any nonzero vector in $\mathbb{R}^p$.

Prove that

$$\max_{\boldsymbol{a} \neq \boldsymbol{0}} \frac{(\boldsymbol{a}'\boldsymbol{b})^2}{\boldsymbol{a}'\boldsymbol{A}\boldsymbol{a}} = \boldsymbol{b}'\boldsymbol{A}^{-1}\boldsymbol{b}$$

and that

$$\frac{(\boldsymbol{a}'\boldsymbol{b})^2}{\boldsymbol{a}'\boldsymbol{A}\boldsymbol{a}} = \boldsymbol{b}'\boldsymbol{A}^{-1}\boldsymbol{b}$$

whenever $\boldsymbol{a} = c\boldsymbol{A}^{-1}\boldsymbol{b}$ for $c \in \mathbb{R} \setminus \{0\}$.

Proof:

From the previous result part (ii), we have

$$\begin{aligned}
&(\boldsymbol{a}'\boldsymbol{b})^2 \leq (\boldsymbol{a}'\boldsymbol{A}\boldsymbol{a})(\boldsymbol{b}'\boldsymbol{A}^{-1}\boldsymbol{b}) \\
\Longrightarrow\ & \frac{(\boldsymbol{a}'\boldsymbol{b})^2}{\boldsymbol{a}'\boldsymbol{A}\boldsymbol{a}} \leq \boldsymbol{b}'\boldsymbol{A}^{-1}\boldsymbol{b} \quad \forall\, \boldsymbol{a} \neq \boldsymbol{0} \\
\Longrightarrow\ & \max_{\boldsymbol{a}\neq\boldsymbol{0}} \frac{(\boldsymbol{a}'\boldsymbol{b})^2}{\boldsymbol{a}'\boldsymbol{A}\boldsymbol{a}} \leq \boldsymbol{b}'\boldsymbol{A}^{-1}\boldsymbol{b}.
\end{aligned}$$

If

$$\boldsymbol{a} = c\boldsymbol{A}^{-1}\boldsymbol{b} \quad \text{for some } c \neq 0,$$

then

$$\begin{aligned} \frac{(\boldsymbol{a}'\boldsymbol{b})^2}{\boldsymbol{a}'\boldsymbol{A}\boldsymbol{a}} &= \frac{((c\boldsymbol{A}^{-1}\boldsymbol{b})'\boldsymbol{b})^2}{(c\boldsymbol{A}^{-1}\boldsymbol{b})'\boldsymbol{A}(c\boldsymbol{A}^{-1}\boldsymbol{b})} \\ &= \frac{(\boldsymbol{b}'\boldsymbol{A}^{-1}\boldsymbol{b})^2}{\boldsymbol{b}'\boldsymbol{A}^{-1}\boldsymbol{b}} \\ &= \boldsymbol{b}'\boldsymbol{A}^{-1}\boldsymbol{b}. \end{aligned}$$

□

Note that if

$$\boldsymbol{b} = \boldsymbol{0},$$

then

$$\max_{\boldsymbol{a} \neq \boldsymbol{0}} \frac{(\boldsymbol{a}'\boldsymbol{b})^2}{\boldsymbol{a}'\boldsymbol{A}\boldsymbol{a}} = \boldsymbol{b}'\boldsymbol{A}^{-1}\boldsymbol{b}$$

holds trivially.

Thus, for any vector $\boldsymbol{b} \in \mathbb{R}^p$ and for any symmetric and positive definite $\underset{p \times p}{\boldsymbol{A}}$,

$$\max_{\boldsymbol{a} \neq \boldsymbol{0}} \frac{(\boldsymbol{a}'\boldsymbol{b})^2}{\boldsymbol{a}'\boldsymbol{A}\boldsymbol{a}} = \boldsymbol{b}'\boldsymbol{A}^{-1}\boldsymbol{b}.$$