The Bonferroni Confidence Rectangle

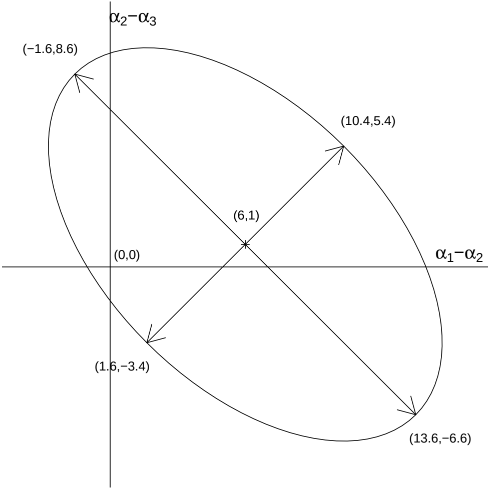
Suppose

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} \sim N(\mathbf{0}, \sigma^2 \boldsymbol{I}).$$

Let $\boldsymbol{c}_1'\boldsymbol{\beta}, \ldots, \boldsymbol{c}_m'\boldsymbol{\beta}$ be m estimable functions.

We want to find an m-dimensional rectangle that contains $\begin{bmatrix} \boldsymbol{c}_1'\boldsymbol{\beta} \\ \vdots \\ \boldsymbol{c}_m'\boldsymbol{\beta} \end{bmatrix}$ with probability at least $1 - \alpha$.

We may prefer a rectangular confidence region to an ellipsoidal region for ease of interpretation.



Let $E_j(\alpha)$ denote the event

$$\boldsymbol{c}_j'\boldsymbol{\beta} \in \left(\boldsymbol{c}_j'\hat{\boldsymbol{\beta}} - t_{n-r,\alpha/2}\sqrt{\hat{\sigma}^2\boldsymbol{c}_j'(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{c}_j}, \boldsymbol{c}_j'\hat{\boldsymbol{\beta}} + t_{n-r,\alpha/2}\sqrt{\hat{\sigma}^2\boldsymbol{c}_j'(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{c}_j}\right).$$

From our previous result, we know

$$\mathbb{P}[E_j(\alpha)] = 1 - \alpha \quad \forall\, j = 1, \ldots, m.$$

Note that

$$\mathbb{P}(\cap_{j=1}^{m} E_j(\alpha)) \leq 1 - \alpha$$

with strict inequality in most cases.

(An exception: $\boldsymbol{c}_1 = \cdots = \boldsymbol{c}_m$).

We will show that

$$\mathbb{P}(\cap_{j=1}^{m} E_j(\alpha/m)) \geq 1-\alpha.$$

$\therefore (l_1, u_1) \times \cdots \times (l_m, u_m)$ is a rectangular confidence region with the desired coverage probability, where

$$\begin{aligned} l_j &= \boldsymbol{c}_j'\hat{\boldsymbol{\beta}} - t_{n-r,\frac{\alpha}{2m}}\sqrt{\hat{\sigma}^2\boldsymbol{c}_j'(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{c}_j} \quad \text{and} \\ u_j &= \boldsymbol{c}_j'\hat{\boldsymbol{\beta}} + t_{n-r,\frac{\alpha}{2m}}\sqrt{\hat{\sigma}^2\boldsymbol{c}_j'(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{c}_j} \quad \forall\, j = 1, \ldots, m. \end{aligned}$$

We say the Bonferroni intervals

$$(l_j, u_j) \quad j = 1, \ldots, m$$

have simultaneous coverage probability at least $1 - \alpha \because$

$$\mathbb{P}[\boldsymbol{c}_j'\boldsymbol{\beta} \in (l_j, u_j) \quad \forall\, j = 1, \ldots, m] \geq 1 - \alpha.$$

A collection of subsets of a sample space $\mathcal{S}$ is called a sigma algebra, denoted $\mathcal{B}$, if

(i) $\emptyset \in \mathcal{B}$

(ii) $A \in \mathcal{B} \Longrightarrow A^c \in \mathcal{B}$

(iii) $A_1, A_2, \ldots \in \mathcal{B} \Longrightarrow \cup_{j=1}^{\infty} A_j \in \mathcal{B}$.

Given a sample space $\mathcal{S}$ and an associated sigma algebra $\mathcal{B}$, a <u>probability function</u> with domain $\mathcal{B}$ satisfies

(i) $\mathbb{P}(A) \geq 0 \quad \forall\, A \in \mathcal{B}$

(ii) $\mathbb{P}(\mathcal{S}) = 1$

(iii) $A_1, A_2, \ldots \in \mathcal{B}$ pairwise disjoint $\Longrightarrow \mathbb{P}(\cup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \mathbb{P}(A_j)$.

It follows that $\forall$ set $A \in \mathcal{B}$,

$$\mathbb{P}(A) + \mathbb{P}(A^c) = \mathbb{P}(A \cup A^c) = \mathbb{P}(\mathcal{S}) = 1.$$

$\therefore \forall\, A \in \mathcal{B}, \mathbb{P}(A) = 1 - \mathbb{P}(A^c)$.

Boole's inequality:

For any sets $A_1, \ldots, A_m \in \mathcal{B}$,

$$\mathbb{P}(\cup_{j=1}^m A_j) \leq \sum_{j=1}^m \mathbb{P}(A_j).$$

Proof:

Let

$$\begin{aligned}
B_1 &= A_1 \\
B_2 &= A_1^c \cap A_2 \\
B_3 &= A_1^c \cap A_2^c \cap A_3 \\
&\vdots \\
B_m &= A_1^c \cap \cdots \cap A_{m-1}^c \cap A_m.
\end{aligned}$$

Then

$$B_j \subseteq A_j \quad \forall\, j = 1, \ldots, m.$$

$B_1, \ldots, B_m$ are pairwise disjoint, and

$$\cup_{j=1}^m B_j = \cup_{j=1}^m A_j.$$

Thus,

$$\begin{aligned} \mathbb{P}(\cup_{j=1}^m A_j) &= \mathbb{P}(\cup_{j=1}^m B_j) \\ &= \sum_{j=1}^m \mathbb{P}(B_j) \\ &\leq \sum_{j=1}^m \mathbb{P}(A_j). \end{aligned}$$

$\square$

Bonferroni's Inequality:

For any sets $A_1, \ldots, A_m \in \mathcal{B}$

$$\mathbb{P}(\cap_{j=1}^m A_j) \geq 1 - \sum_{j=1}^m \mathbb{P}(A_j^c).$$

Prove this inequality using basic results from probability theory.

Proof:

$$\begin{aligned}
\mathbb{P}(\cap_{j=1}^{m} A_j) &= 1 - \mathbb{P}((\cap_{j=1}^{m} A_j)^c) \\
&= 1 - \mathbb{P}(\cup_{j=1}^{m} A_j^c) \\
&\geq 1 - \sum_{j=1}^{m} \mathbb{P}(A_j^c).
\end{aligned}$$

Now prove that

$$\mathbb{P}[\cap_{j=1}^{m} E_j(\alpha/m)] \geq 1 - \alpha.$$

Proof:

$$
\begin{aligned}
\mathbb{P}[\cap_{j=1}^{m} E_j(\alpha/m)] &\geq 1 - \sum_{j=1}^{m} \mathbb{P}[E_j(\alpha/m)^c] \\
&= 1 - \sum_{j=1}^{m} (1 - \mathbb{P}[E_j(\alpha/m)]) \\
&= 1 - \sum_{j=1}^{m} (1 - (1 - \alpha/m)) \\
&= 1 - \sum_{j=1}^{m} \alpha/m \\
&= 1 - m\alpha/m = 1 - \alpha.
\end{aligned}
$$

$\square$

Under what circumstances does the Bonferroni confidence rectangle

$(l_1, u_1) \times \cdots \times (l_m, u_m)$ give exact $1 - \alpha$ coverage of $\begin{bmatrix} \boldsymbol{c}_1'\boldsymbol{\beta} \\ \vdots \\ \boldsymbol{c}_m'\boldsymbol{\beta} \end{bmatrix}$?

Looking back at the proof of Bonferroni's inequality, we have

$$\mathbb{P}(\cap_{j=1}^m A_j) = 1 - \sum_{j=1}^m \mathbb{P}(A_j^c)$$

iff

$$\mathbb{P}(\cup_{j=1}^m A_j^c) = \sum_{j=1}^m \mathbb{P}(A_j^c).$$

$$\mathbb{P}(\cup_{j=1}^{m} A_j^c) = \sum_{j=1}^{m} \mathbb{P}(A_j^c)$$
$$\iff \mathbb{P}(A_i^c \cap A_j^c) = 0 \quad \forall\, i \neq j.$$

Thus, we have

$$\mathbb{P}[\cap_{j=1}^{m} E_j(\alpha/m)] = 1 - \alpha$$
$$\iff \mathbb{P}[E_i(\alpha/m)^c \cap E_j(\alpha/m)^c] = 0 \quad \forall\, i \neq j.$$

This says that the Bonferroni confidence rectangle is exact iff the failure of any one interval to cover its target parameter implies that all other intervals cover their target parameters with probability 1; i.e.,

$$\boldsymbol{c}_j'\boldsymbol{\beta} \notin (l_j, u_j) \Longrightarrow \boldsymbol{c}_i'\boldsymbol{\beta} \in (l_i, u_i) \quad \forall\ i \neq j.$$

Thus, in practice

$$\mathbb{P}[\cap_{j=1}^{m} E_j(\alpha/m)] > 1 - \alpha.$$

Note that if

$$E_1(\alpha/m), \ldots, E_m(\alpha/m)$$

are independent events, then

$$\begin{aligned} \mathbb{P}[\cap_{j=1}^m E_j(\alpha/m)] &= \prod_{j=1}^m \mathbb{P}[E_j(\alpha/m)] \\ &= \prod_{j=1}^m (1 - \alpha/m) = (1 - \alpha/m)^m \\ &> 1 - \alpha \quad \text{for } m > 1. \end{aligned}$$

Under independence,

$$\begin{aligned}
\mathbb{P}[\cap_{j=1}^m E_j(1-(1-\alpha)^{1/m})] &= \prod_{j=1}^m \mathbb{P}[E_j(1-(1-\alpha)^{1/m})] \\
&= [(1-\alpha)^{1/m}]^m \\
&= 1-\alpha.
\end{aligned}$$