

# General Linear Test of a General Linear Hypothesis

Suppose the NTGMM holds so that

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon},$$

where

$$\boldsymbol{\varepsilon} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}).$$

Suppose  $C$  is a known  $q \times p$  matrix and  $d$  is a known  $q \times 1$  vector.

The general linear hypothesis

$$H_0 : C\beta = d$$

is testable if  $rank(C) = q$  and each component of  $C\beta$  is estimable.

Suppose  $\mathbf{A}$  of rank  $s$ .  
 $p \times m$

Can  $H_0 : \boldsymbol{\beta} \in \mathcal{C}(\mathbf{A})$  be written as a testable general linear hypothesis?

Suppose

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk}, \quad i = 1, 2; j = 1, 2; k = 1, \dots, n_{ij}.$$

Let  $\beta = \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \gamma_{11} \\ \gamma_{12} \\ \gamma_{21} \\ \gamma_{22} \end{bmatrix}$ .

Write a testable general linear hypothesis for “no interaction.”

Suppose

$$H_0 : C\beta = d$$

is testable.

Find the distribution of the BLUE of  $C\beta$ .

Find the distribution of

$$(\mathbf{C}\hat{\boldsymbol{\beta}} - \mathbf{d})'(\sigma^2\mathbf{C}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{C}')^{-1}(\mathbf{C}\hat{\boldsymbol{\beta}} - \mathbf{d}).$$

Show that  $C\hat{\beta}$  and SSE are independent.



Now note that independence of  $C\hat{\beta}$  and SSE  $\implies$

$$(C\hat{\beta} - d)'(\sigma^2 C(X'X)^{-1}C')^{-1}(C\hat{\beta} - d)$$

and

$$\frac{\text{SSE}}{\sigma^2(n-r)} = \frac{\mathbf{y}'(\mathbf{I} - \mathbf{P}_X)\mathbf{y}}{\sigma^2(n-r)} = \frac{\hat{\sigma}^2}{\sigma^2}$$

are independent.

We have previously shown that

$$\frac{\text{SSE}}{\sigma^2} \sim \chi_{n-r}^2.$$

Thus,

$$\frac{(n-r)\hat{\sigma}^2}{\sigma^2} \sim \chi_{n-r}^2$$

and

$$\frac{\hat{\sigma}^2}{\sigma^2} \sim \chi_{n-r}^2 / (n-r).$$

It follows that

$$\begin{aligned} & \frac{(\mathbf{C}\hat{\boldsymbol{\beta}} - \mathbf{d})'(\sigma^2\mathbf{C}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{C}')^{-1}(\mathbf{C}\hat{\boldsymbol{\beta}} - \mathbf{d})/q}{\hat{\sigma}^2/\sigma^2} \\ &= \frac{(\mathbf{C}\hat{\boldsymbol{\beta}} - \mathbf{d})'(\mathbf{C}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{C}')^{-1}(\mathbf{C}\hat{\boldsymbol{\beta}} - \mathbf{d})}{q\hat{\sigma}^2} \equiv F \\ &\sim F_{q,n-r}(\phi), \end{aligned}$$

where

$$\phi = \frac{1}{2\sigma^2}(\mathbf{C}\boldsymbol{\beta} - \mathbf{d})'(\mathbf{C}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{C}')^{-1}(\mathbf{C}\boldsymbol{\beta} - \mathbf{d})$$

as defined previously.

We can use  $F$  to test

$$\begin{aligned}H_0 : \phi = 0 &\iff H_0 : \mathbf{C}\boldsymbol{\beta} - \mathbf{d} = \mathbf{0} \\ &(\because (\mathbf{C}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{C}')^{-1} \text{ is PD.}) \\ &\iff H_0 : \mathbf{C}\boldsymbol{\beta} = \mathbf{d}.\end{aligned}$$

To test

$$H_0 : C\beta = d$$

at level  $\alpha$ , we reject  $H_0$  iff

$$F \geq F_{q,n-r,\alpha}$$

where  $F_{q,n-r,\alpha}$  is the upper  $\alpha$  quantile of the  $F_{q,n-r}$  distribution.

By Result 5.13, the power of the test is a strictly increasing function of  $\phi$ .

Now suppose that

$$H_0 : \mathbf{c}'\boldsymbol{\beta} = d$$

is testable.

By arguments analogous to the previous  $F$  case, it is straightforward to show that

$$t \equiv \frac{\mathbf{c}'\hat{\boldsymbol{\beta}} - d}{\sqrt{\hat{\sigma}^2 \mathbf{c}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{c}}} \sim t_{n-r} \left( \frac{\mathbf{c}'\boldsymbol{\beta} - d}{\sqrt{\sigma^2 \mathbf{c}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{c}}} \right).$$

We can conduct tests of

$$H_0 : \mathbf{c}'\boldsymbol{\beta} = d \quad \text{against} \quad H_{A_1} : \mathbf{c}'\boldsymbol{\beta} < d$$
$$H_{A_2} : \mathbf{c}'\boldsymbol{\beta} > d, \quad \text{or}$$
$$H_A : \mathbf{c}'\boldsymbol{\beta} \neq d$$

by comparing the observed value of  $t$  to the  $t_{n-r}$  distribution.

Returning to the  $F$ -test of testable GLH

$$H_0 : C\beta = d,$$

note that there are multiple ways to express the same null hypothesis.



For example, suppose

$$y_{ij} = \mu_i + \varepsilon_{ij} \quad (i = 1, 2, 3; j = 1, \dots, n_i).$$

Find different matrices  $\mathbf{C}_1$ ,  $\mathbf{C}_2$  and  $\mathbf{C}_3 \ni$

$$\mathbf{C}_k \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = \mathbf{0} \iff \mu_1 = \mu_2 = \mu_3 \quad \forall k = 1, 2, 3.$$

$$\mathbf{C}_1 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \quad \mathbf{C}_1\boldsymbol{\mu} = \begin{bmatrix} \mu_1 - \mu_2 \\ \mu_1 - \mu_3 \end{bmatrix}$$

$$\mathbf{C}_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad \mathbf{C}_2\boldsymbol{\mu} = \begin{bmatrix} \mu_1 - \mu_2 \\ \mu_2 - \mu_3 \end{bmatrix}$$

$$\mathbf{C}_3 = \begin{bmatrix} 1 & -1 & 0 \\ 1/2 & 1/2 & -1 \end{bmatrix}, \quad \mathbf{C}_3\boldsymbol{\mu} = \begin{bmatrix} \mu_1 - \mu_2 \\ \frac{\mu_1 + \mu_2}{2} - \mu_3 \end{bmatrix}.$$

Suppose

$$H_{01} : \mathbf{C}_1\boldsymbol{\beta} = \mathbf{d}_1 \quad \text{and} \quad H_{02} : \mathbf{C}_2\boldsymbol{\beta} = \mathbf{d}_2$$

are both testable and

$$\begin{aligned} \mathcal{S}_1 &\equiv \{\boldsymbol{\beta} : \mathbf{C}_1\boldsymbol{\beta} = \mathbf{d}_1\} \\ &= \{\boldsymbol{\beta} : \mathbf{C}_2\boldsymbol{\beta} = \mathbf{d}_2\} \equiv \mathcal{S}_2. \end{aligned}$$

Show that the  $F$ -test of  $H_{01}$  is the same as the  $F$ -test of  $H_{02}$ .

Recall that

$$\mathbf{X}' = \mathbf{X}'\mathbf{P}_X = \mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'.$$

Thus,  $\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}$  is a GI of  $\mathbf{X}' \quad \forall \mathbf{X}'$ .

It follows that  $\mathbf{C}'_k(\mathbf{C}_k\mathbf{C}'_k)^{-}$  is a GI of  $\mathbf{C}_k$ . Thus,

$$\mathcal{S}_k = \{\mathbf{C}'_k(\mathbf{C}_k\mathbf{C}'_k)^{-}\mathbf{d}_k + (\mathbf{I} - \mathbf{C}'_k(\mathbf{C}_k\mathbf{C}'_k)^{-}\mathbf{C}_k)\mathbf{z} : \mathbf{z} \in \mathbb{R}^p\}$$

for  $k = 1, 2$ .

Because  $\mathcal{S}_1 = \mathcal{S}_2$ ,  $\mathbf{C}_1$  times any element of  $\mathcal{S}_2$  equal  $\mathbf{d}_1$ ; i.e.,

$$\boldsymbol{\beta} \in \mathcal{S}_2 \implies \mathbf{C}_1 \boldsymbol{\beta} = \mathbf{d}_1.$$

Thus

$$\mathbf{C}_1 [\mathbf{C}'_2 (\mathbf{C}_2 \mathbf{C}'_2)^{-} \mathbf{d}_2 + (\mathbf{I} - \mathbf{C}'_2 (\mathbf{C}_2 \mathbf{C}'_2)^{-} \mathbf{C}_2) \mathbf{z}] = \mathbf{d}_1 \quad \forall \mathbf{z} \in \mathbb{R}^p$$

$$\implies \mathbf{C}_1 \mathbf{C}'_2 (\mathbf{C}_2 \mathbf{C}'_2)^{-} \mathbf{d}_2 - \mathbf{d}_1 + (\mathbf{C}_1 - \mathbf{C}_1 \mathbf{C}'_2 (\mathbf{C}_2 \mathbf{C}'_2)^{-} \mathbf{C}_2) \mathbf{z} = \mathbf{0} \quad \forall \mathbf{z} \in \mathbb{R}^p$$

$$\implies \mathbf{C}_1 = \mathbf{C}_1 \mathbf{C}'_2 (\mathbf{C}_2 \mathbf{C}'_2)^{-} \mathbf{C}_2 \quad \text{and}$$

$$\mathbf{C}_1 \mathbf{C}'_2 (\mathbf{C}_2 \mathbf{C}'_2)^{-} \mathbf{d}_2 = \mathbf{d}_1 \quad \text{by Result A.8.}$$

Now

$$\begin{aligned}C_1 &= C_1 C_2' (C_2 C_2')^{-1} C_2 \\ \implies C_1 &= C_1 P_{C_2'} \\ \implies C_1' &= P_{C_2'} C_1' \\ \implies \mathcal{C}(C_1') &\subseteq \mathcal{C}(P_{C_2'}) = \mathcal{C}(C_2').\end{aligned}$$

Repeating the entire argument with the roles of  $C_1$  and  $C_2$  reversed gives

$$\mathcal{C}(C_2') \subseteq \mathcal{C}(C_1')$$

so that

$$\mathcal{C}(C_2') = \mathcal{C}(C_1').$$

Because

$$\mathbf{C}_1\boldsymbol{\beta} = \mathbf{d}_1 \quad \text{and} \quad \mathbf{C}_2\boldsymbol{\beta} = \mathbf{d}_2$$

are both testable,  $\mathbf{C}'_1$  and  $\mathbf{C}'_2$  are both full-column rank =  $q$ .

$\therefore \exists$  a unique nonsingular matrix  $\mathbf{B} \ni_{q \times q}$

$$\mathbf{C}'_1 = \mathbf{C}'_2\mathbf{B}.$$

We have previously shown that

$$\mathbf{C}'_1 = \mathbf{P}_{\mathbf{C}'_2}\mathbf{C}'_1 = \mathbf{C}'_2[(\mathbf{C}_2\mathbf{C}'_2)^-]'\mathbf{C}_2\mathbf{C}'_1.$$

Thus,

$$\mathbf{B} = [(\mathbf{C}_2\mathbf{C}'_2)^-]'\mathbf{C}_2\mathbf{C}'_1 \quad \text{and} \quad \mathbf{B}' = \mathbf{C}_1\mathbf{C}'_2(\mathbf{C}_2\mathbf{C}'_2)^-.$$

We have also shown previously that

$$\mathbf{C}_1 \mathbf{C}_2' (\mathbf{C}_2 \mathbf{C}_2')^{-} \mathbf{d}_2 = \mathbf{d}_1 \implies \mathbf{B}' \mathbf{d}_2 = \mathbf{d}_1.$$



Now consider the quadratic form

$$\begin{aligned} & (\mathbf{C}_1\mathbf{b} - \mathbf{d}_1)'(\mathbf{C}_1(\mathbf{X}'\mathbf{X})^{-1}\mathbf{C}_1')^{-1}(\mathbf{C}_1\mathbf{b} - \mathbf{d}_1) \\ &= (\mathbf{B}'\mathbf{C}_2\mathbf{b} - \mathbf{B}'\mathbf{d}_2)'(\mathbf{B}'\mathbf{C}_2(\mathbf{X}'\mathbf{X})^{-1}\mathbf{C}_2'\mathbf{B})^{-1}(\mathbf{B}'\mathbf{C}_2\mathbf{b} - \mathbf{B}'\mathbf{d}_2) \\ &= (\mathbf{C}_2\mathbf{b} - \mathbf{d}_2)'\mathbf{B}\mathbf{B}^{-1}(\mathbf{C}_2(\mathbf{X}'\mathbf{X})^{-1}\mathbf{C}_2')^{-1}(\mathbf{B}')^{-1}\mathbf{B}'(\mathbf{C}_2\mathbf{b} - \mathbf{d}_2) \\ &= (\mathbf{C}_2\mathbf{b} - \mathbf{d}_2)'(\mathbf{C}_2(\mathbf{X}'\mathbf{X})^{-1}\mathbf{C}_2')^{-1}(\mathbf{C}_2\mathbf{b} - \mathbf{d}_2). \end{aligned}$$

This is true for  $\forall \mathbf{b} \in \mathbb{R}^p$  including  $\hat{\beta}$  and the true parameter vector  $\beta$ .

∴ the  $F$  statistics for testing

$$H_{01} : \mathbf{C}_1\boldsymbol{\beta} = \mathbf{d}_1 \quad \text{and} \quad H_{02} : \mathbf{C}_2\boldsymbol{\beta} = \mathbf{d}_2$$

are identical, as are the noncentrality parameters associated with those  $F$  statistics.