General Linear Test of a General Linear Hypothesis

Suppose the NTGMM holds so that

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon},$$

where

$$\boldsymbol{\varepsilon} \sim N(\mathbf{0}, \sigma^2 \boldsymbol{I}).$$

Suppose $\boldsymbol{C}$ is a known $q \times p$ matrix and $\boldsymbol{d}$ is a known $q \times 1$ vector.

The general linear hypothesis

$$H_0 : \boldsymbol{C\beta} = \boldsymbol{d}$$

is testable if $rank(\boldsymbol{C}) = q$ and each component of $\boldsymbol{C\beta}$ is estimable.

Suppose $\underset{p \times m}{\boldsymbol{A}}$ of rank s.

Can $H_0 : \boldsymbol{\beta} \in \mathcal{C}(\boldsymbol{A})$ be written as a testable general linear hypothesis?

$$\begin{aligned}\boldsymbol{\beta} \in \mathcal{C}(\boldsymbol{A}) &\iff \boldsymbol{P_A}\boldsymbol{\beta} = \boldsymbol{\beta} \\ &\iff \boldsymbol{\beta} - \boldsymbol{P_A}\boldsymbol{\beta} = \mathbf{0} \\ &\iff (\boldsymbol{I} - \boldsymbol{P_A})\boldsymbol{\beta} = \mathbf{0} \\ &\iff \begin{bmatrix} \boldsymbol{w}_1' \\ \vdots \\ \boldsymbol{w}_{p-s}' \end{bmatrix} \boldsymbol{\beta} = \mathbf{0},\end{aligned}$$

where $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{p-s}$ form a basis for $\mathcal{C}((\boldsymbol{I} - \boldsymbol{P_A})') = \mathcal{C}(\boldsymbol{I} - \boldsymbol{P_A})$.

Suppose

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk}, \quad i = 1, 2;\ j = 1, 2;\ k = 1, \ldots, n_{ij}.$$

Let $\boldsymbol{\beta} = \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \gamma_{11} \\ \gamma_{12} \\ \gamma_{21} \\ \gamma_{22} \end{bmatrix}.$

Write a testable general linear hypothesis for "no interaction."

In this case, no interaction means

$$
\begin{aligned}
& E(y_{11k}) - E(y_{12k}) = E(y_{21k}) - E(y_{22k}) \\
\iff & E(y_{11k}) - E(y_{12k}) - E(y_{21k}) + E(y_{22k}) = 0 \\
\iff & \mu + \alpha_1 + \beta_1 + \gamma_{11} \\
& - (\mu + \alpha_1 + \beta_2 + \gamma_{12}) \\
& - (\mu + \alpha_2 + \beta_1 + \gamma_{21}) \\
& + (\mu + \alpha_2 + \beta_2 + \gamma_{22}) = 0 \\
\iff & \gamma_{11} - \gamma_{12} - \gamma_{21} + \gamma_{22} = 0.
\end{aligned}
$$

Thus,

$$H_0 : \boldsymbol{C}\boldsymbol{\beta} = \boldsymbol{0}$$

is testable GLH of no interaction if

$$\boldsymbol{C} = [0, 0, 0, 0, 0, 1, -1, -1, 1]$$

because $\boldsymbol{C}$ is $1 \times p$ is of rank 1 and

$$\begin{aligned} \boldsymbol{C}\boldsymbol{\beta} &= \gamma_{11} - \gamma_{12} - \gamma_{21} + \gamma_{22} \\ &= E(y_{111}) - E(y_{121}) - E(y_{211}) + E(y_{221}) \end{aligned}$$

is estimable as a LC of elements of $E(\boldsymbol{y})$.

Suppose

$$H_0 : \boldsymbol{C\beta} = \boldsymbol{d}$$

is testable.

Find the distribution of the BLUE of $\boldsymbol{C\beta}$.

$$\boldsymbol{C}\hat{\boldsymbol{\beta}} = \boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{y} \sim N(\boldsymbol{C}\boldsymbol{\beta}, \sigma^2\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{C}')$$

where $\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{C}'$ is a PD $q \times q$ matrix of rank q based on previous results.

Find the distribution of

$$(\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{d})'(\sigma^2 \boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{C}')^{-1}(\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{d}).$$

By Result 5.10, the distribution is $\chi^2_q(\phi)$, where

$$\begin{aligned}\phi &= \frac{1}{2}(\boldsymbol{C}\boldsymbol{\beta} - \boldsymbol{d})'(\sigma^2\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{C}')^{-1}(\boldsymbol{C}\boldsymbol{\beta} - \boldsymbol{d}) \\ &= \frac{1}{2\sigma^2}(\boldsymbol{C}\boldsymbol{\beta} - \boldsymbol{d})'(\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{C}')^{-1}(\boldsymbol{C}\boldsymbol{\beta} - \boldsymbol{d}).\end{aligned}$$

Show that $\boldsymbol{C}\hat{\boldsymbol{\beta}}$ and SSE are independent.

$$
\begin{bmatrix} \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{\varepsilon}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{P_X y} \\ (\boldsymbol{I} - \boldsymbol{P_X})\boldsymbol{y} \end{bmatrix} = \begin{bmatrix} \boldsymbol{P_X} \\ \boldsymbol{I} - \boldsymbol{P_X} \end{bmatrix} \boldsymbol{y}
$$

$$
\sim N\left(\begin{bmatrix} \boldsymbol{P_X} \\ \boldsymbol{I} - \boldsymbol{P_X} \end{bmatrix} \boldsymbol{X\beta}, \begin{bmatrix} \boldsymbol{P_X} \\ \boldsymbol{I} - \boldsymbol{P_X} \end{bmatrix} \sigma^2 \boldsymbol{I} \begin{bmatrix} \boldsymbol{P_X} & \boldsymbol{I} - \boldsymbol{P_X} \end{bmatrix} \right)
$$

$$
\sim N\left(\begin{bmatrix} \boldsymbol{X\beta} \\ \boldsymbol{0} \end{bmatrix}, \sigma^2 \begin{bmatrix} \boldsymbol{P_X} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} - \boldsymbol{P_X} \end{bmatrix} \right).
$$

Thus, $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{\varepsilon}}$ are independent.

$\boldsymbol{C}\boldsymbol{\beta}$ estimable $\Longrightarrow \exists\, \boldsymbol{A} \ni \boldsymbol{C} = \boldsymbol{A}\boldsymbol{X}$.

$$
\begin{aligned}
\therefore \boldsymbol{C}\hat{\boldsymbol{\beta}} &= \boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{y} \\
&= \boldsymbol{A}\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{y} \\
&= \boldsymbol{A}\boldsymbol{P}_{\boldsymbol{X}}\boldsymbol{y} \\
&= \boldsymbol{A}\hat{\boldsymbol{y}}.
\end{aligned}
$$

$$\text{SSE} = \hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}}.$$

$\because$ $\boldsymbol{C}\hat{\boldsymbol{\beta}}$ is a function of only $\hat{\boldsymbol{y}}$ and SSE a function of only $\hat{\boldsymbol{\varepsilon}}$, $\boldsymbol{C}\hat{\boldsymbol{\beta}}$ and SSE are independent.

We could alternatively have used Result 5.16:

$$\begin{aligned}
&\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'(\sigma^2\boldsymbol{I})(\boldsymbol{I}-\boldsymbol{P_X})\\
&=\sigma^2\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'(\boldsymbol{I}-\boldsymbol{P_X})\\
&=\sigma^2\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-}(\boldsymbol{X}'-\boldsymbol{X}'\boldsymbol{P_X})\\
&=\sigma^2\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-}(\boldsymbol{X}'-\boldsymbol{X}')=\boldsymbol{0}.
\end{aligned}$$

Now note that independence of $\boldsymbol{C}\hat{\boldsymbol{\beta}}$ and $\text{SSE} \Longrightarrow$

$$(\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{d})'(\sigma^2 \boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^- \boldsymbol{C}')^{-1}(\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{d})$$

and

$$\frac{\text{SSE}}{\sigma^2(n-r)} = \frac{\boldsymbol{y}'(\boldsymbol{I} - \boldsymbol{P_X})\boldsymbol{y}}{\sigma^2(n-r)} = \frac{\hat{\sigma}^2}{\sigma^2}$$

are independent.

We have previously shown that

$$\frac{\text{SSE}}{\sigma^2} \sim \chi^2_{n-r}.$$

Thus,

$$\frac{(n-r)\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-r}$$

and

$$\frac{\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-r}/(n-r).$$

It follows that

$$\frac{(\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{d})'(\sigma^2 \boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^- \boldsymbol{C}')^{-1}(\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{d})/q}{\hat{\sigma}^2/\sigma^2}$$

$$= \frac{(\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{d})'(\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^- \boldsymbol{C}')^{-1}(\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{d})}{q\hat{\sigma}^2} \equiv F$$

$$\sim F_{q,n-r}(\phi),$$

where

$$\phi = \frac{1}{2\sigma^2}(\boldsymbol{C\beta} - \boldsymbol{d})'(\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{C}')^{-1}(\boldsymbol{C\beta} - \boldsymbol{d})$$

as defined previously.

We can use F to test

$$H_0 : \phi = 0 \iff H_0 : \boldsymbol{C\beta} - \boldsymbol{d} = \boldsymbol{0}$$
$$(\because (\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{C}')^{-1} \text{ is PD.})$$
$$\iff H_0 : \boldsymbol{C\beta} = \boldsymbol{d}.$$

To test

$$H_0 : \boldsymbol{C\beta} = \boldsymbol{d}$$

at level α, we reject H_0 iff

$$F \geq F_{q,n-r,\alpha}$$

where $F_{q,n-r,\alpha}$ is the upper α quantile of the $F_{q,n-r}$ distribution.

By Result 5.13, the power of the test is a strictly increasing function of ϕ.

Now suppose that

$$H_0 : \boldsymbol{c}'\boldsymbol{\beta} = d$$

is testable.

By arguments analogous to the previous F case, it is straightforward to show that

$$t \equiv \frac{\boldsymbol{c}'\hat{\boldsymbol{\beta}} - d}{\sqrt{\hat{\sigma}^2 \boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{c}}} \sim t_{n-r}\left(\frac{\boldsymbol{c}'\boldsymbol{\beta} - d}{\sqrt{\sigma^2 \boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{c}}}\right).$$

We can conduct tests of

$$H_0 : \boldsymbol{c}'\boldsymbol{\beta} = d \quad \text{against} \quad \begin{aligned} &H_{A_1} : \boldsymbol{c}'\boldsymbol{\beta} < d \\ &H_{A_2} : \boldsymbol{c}'\boldsymbol{\beta} > d, \quad \text{or} \\ &H_A : \boldsymbol{c}'\boldsymbol{\beta} \neq d \end{aligned}$$

by comparing the observed value of t to the t_{n-r} distribution.

Returning to the F-test of testable GLH

$$H_0 : \boldsymbol{C}\boldsymbol{\beta} = \boldsymbol{d},$$

note that there are multiple ways to express the same null hypothesis.

For example, suppose

$$y_{ij} = \mu_i + \varepsilon_{ij} \quad (i = 1, 2, 3;\ j = 1, \ldots, n_i).$$

Find different matrices $\boldsymbol{C}_1, \boldsymbol{C}_2$ and $\boldsymbol{C}_3 \ni$

$$\boldsymbol{C}_k \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = \mathbf{0} \iff \mu_1 = \mu_2 = \mu_3 \quad \forall\, k = 1, 2, 3.$$

$$\boldsymbol{C}_1 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \qquad \boldsymbol{C}_1\mu = \begin{bmatrix} \mu_1 - \mu_2 \\ \mu_1 - \mu_3 \end{bmatrix}$$

$$\boldsymbol{C}_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \qquad \boldsymbol{C}_2\mu = \begin{bmatrix} \mu_1 - \mu_2 \\ \mu_2 - \mu_3 \end{bmatrix}$$

$$\boldsymbol{C}_3 = \begin{bmatrix} 1 & -1 & 0 \\ 1/2 & 1/2 & -1 \end{bmatrix}, \qquad \boldsymbol{C}_3\mu = \begin{bmatrix} \mu_1 - \mu_2 \\ \frac{\mu_1+\mu_2}{2} - \mu_3 \end{bmatrix}.$$

Suppose

$$H_{01} : \boldsymbol{C}_1\boldsymbol{\beta} = \boldsymbol{d}_1 \quad \text{and} \quad H_{02} : \boldsymbol{C}_2\boldsymbol{\beta} = \boldsymbol{d}_2$$

are both testable and

$$\begin{aligned} \mathcal{S}_1 &\equiv \{\boldsymbol{\beta} : \boldsymbol{C}_1\boldsymbol{\beta} = \boldsymbol{d}_1\} \\ &= \{\boldsymbol{\beta} : \boldsymbol{C}_2\boldsymbol{\beta} = \boldsymbol{d}_2\} \equiv \mathcal{S}_2. \end{aligned}$$

Show that the F-test of H_{01} is the same as the F-test of H_{02}.

Recall that

$$\boldsymbol{X}' = \boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{X}} = \boldsymbol{X}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'.$$

Thus, $\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}$ is a GI of $\boldsymbol{X}' \quad \forall\, \boldsymbol{X}'$.

It follows that $\boldsymbol{C}_k'(\boldsymbol{C}_k\boldsymbol{C}_k')^{-}$ is a GI of $\boldsymbol{C}_k$. Thus,

$$\mathcal{S}_k = \{\boldsymbol{C}_k'(\boldsymbol{C}_k\boldsymbol{C}_k')^{-}\boldsymbol{d}_k + (\boldsymbol{I} - \boldsymbol{C}_k'(\boldsymbol{C}_k\boldsymbol{C}_k')^{-}\boldsymbol{C}_k)\boldsymbol{z} : \boldsymbol{z} \in \mathbb{R}^p\}$$

for $k = 1, 2$.

Because $\mathcal{S}_1 = \mathcal{S}_2$, $\boldsymbol{C}_1$ times any element of $\mathcal{S}_2$ equal $\boldsymbol{d}_1$; i.e.,

$$\boldsymbol{\beta} \in \mathcal{S}_2 \Longrightarrow \boldsymbol{C}_1\boldsymbol{\beta} = \boldsymbol{d}_1.$$

Thus

$$\begin{aligned}
&\boldsymbol{C}_1[\boldsymbol{C}_2'(\boldsymbol{C}_2\boldsymbol{C}_2')^-\boldsymbol{d}_2 + (\boldsymbol{I} - \boldsymbol{C}_2'(\boldsymbol{C}_2\boldsymbol{C}_2')^-\boldsymbol{C}_2)\boldsymbol{z}] = \boldsymbol{d}_1 \quad \forall\, \boldsymbol{z} \in \mathbb{R}^p \\
\Longrightarrow\ &\boldsymbol{C}_1\boldsymbol{C}_2'(\boldsymbol{C}_2\boldsymbol{C}_2')^-\boldsymbol{d}_2 - \boldsymbol{d}_1 + (\boldsymbol{C}_1 - \boldsymbol{C}_1\boldsymbol{C}_2'(\boldsymbol{C}_2\boldsymbol{C}_2')^-\boldsymbol{C}_2)\boldsymbol{z} = \boldsymbol{0} \quad \forall\, \boldsymbol{z} \in \mathbb{R}^p \\
\Longrightarrow\ &\boldsymbol{C}_1 = \boldsymbol{C}_1\boldsymbol{C}_2'(\boldsymbol{C}_2\boldsymbol{C}_2')^-\boldsymbol{C}_2 \quad \text{and} \\
&\boldsymbol{C}_1\boldsymbol{C}_2'(\boldsymbol{C}_2\boldsymbol{C}_2')^-\boldsymbol{d}_2 = \boldsymbol{d}_1 \quad \text{by Result A.8.}
\end{aligned}$$

Now

$$
\begin{aligned}
\boldsymbol{C}_1 &= \boldsymbol{C}_1\boldsymbol{C}_2'(\boldsymbol{C}_2\boldsymbol{C}_2')^-\boldsymbol{C}_2 \\
\Longrightarrow \boldsymbol{C}_1 &= \boldsymbol{C}_1\boldsymbol{P}_{\boldsymbol{C}_2'} \\
\Longrightarrow \boldsymbol{C}_1' &= \boldsymbol{P}_{\boldsymbol{C}_2'}\boldsymbol{C}_1' \\
\Longrightarrow \mathcal{C}(\boldsymbol{C}_1') &\subseteq \mathcal{C}(\boldsymbol{P}_{\boldsymbol{C}_2'}) = \mathcal{C}(\boldsymbol{C}_2').
\end{aligned}
$$

Repeating the entire argument with the roles of $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ reversed gives

$$\mathcal{C}(\boldsymbol{C}_2') \subseteq \mathcal{C}(\boldsymbol{C}_1')$$

so that

$$\mathcal{C}(\boldsymbol{C}_2') = \mathcal{C}(\boldsymbol{C}_1').$$

Because

$$\boldsymbol{C}_1\boldsymbol{\beta} = \boldsymbol{d}_1 \quad \text{and} \quad \boldsymbol{C}_2\boldsymbol{\beta} = \boldsymbol{d}_2$$

are both testable, $\boldsymbol{C}_1'$ and $\boldsymbol{C}_2'$ are both full-column rank = q.

$\therefore \exists$ a unique nonsingular matrix $\underset{q\times q}{\boldsymbol{B}} \ni$

$$\boldsymbol{C}_1' = \boldsymbol{C}_2'\boldsymbol{B}.$$

We have previously shown that

$$\boldsymbol{C}_1' = \boldsymbol{P}_{\boldsymbol{C}_2'}\boldsymbol{C}_1' = \boldsymbol{C}_2'[(\boldsymbol{C}_2\boldsymbol{C}_2')^-]'\boldsymbol{C}_2\boldsymbol{C}_1'.$$

Thus,

$$\boldsymbol{B} = [(\boldsymbol{C}_2\boldsymbol{C}_2')^-]'\boldsymbol{C}_2\boldsymbol{C}_1' \quad \text{and} \quad \boldsymbol{B}' = \boldsymbol{C}_1\boldsymbol{C}_2'(\boldsymbol{C}_2\boldsymbol{C}_2')^-.$$

We have also shown previously that

$$\boldsymbol{C}_1\boldsymbol{C}_2'(\boldsymbol{C}_2\boldsymbol{C}_2')^-\boldsymbol{d}_2 = \boldsymbol{d}_1 \Longrightarrow \boldsymbol{B}'\boldsymbol{d}_2 = \boldsymbol{d}_1.$$

Now consider the quadratic form

$$
\begin{aligned}
&(\boldsymbol{C}_1\boldsymbol{b} - \boldsymbol{d}_1)'(\boldsymbol{C}_1(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{C}_1')^{-1}(\boldsymbol{C}_1\boldsymbol{b} - \boldsymbol{d}_1) \\
&= (\boldsymbol{B}'\boldsymbol{C}_2\boldsymbol{b} - \boldsymbol{B}'\boldsymbol{d}_2)'(\boldsymbol{B}'\boldsymbol{C}_2(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{C}_2'\boldsymbol{B})^{-1}(\boldsymbol{B}'\boldsymbol{C}_2\boldsymbol{b} - \boldsymbol{B}'\boldsymbol{d}_2) \\
&= (\boldsymbol{C}_2\boldsymbol{b} - \boldsymbol{d}_2)'\boldsymbol{B}\boldsymbol{B}^{-1}(\boldsymbol{C}_2(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{C}_2')^{-1}(\boldsymbol{B}')^{-1}\boldsymbol{B}'(\boldsymbol{C}_2\boldsymbol{b} - \boldsymbol{d}_2) \\
&= (\boldsymbol{C}_2\boldsymbol{b} - \boldsymbol{d}_2)'(\boldsymbol{C}_2(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{C}_2')^{-1}(\boldsymbol{C}_2\boldsymbol{b} - \boldsymbol{d}_2).
\end{aligned}
$$

This is true for $\forall\, \boldsymbol{b} \in \mathbb{R}^p$ including $\hat{\boldsymbol{\beta}}$ and the true parameter vector $\boldsymbol{\beta}$.

$\therefore$ the F statistics for testing

$$H_{01} : \boldsymbol{C}_1\boldsymbol{\beta} = \boldsymbol{d}_1 \quad \text{and} \quad H_{02} : \boldsymbol{C}_2\boldsymbol{\beta} = \boldsymbol{d}_2$$

are identical, as are the noncentrality parameters associated with those F statistics.