Distributions of Quadratic Forms

Under the Normal Theory GMM (NTGMM),

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon}, \quad \text{where} \quad \boldsymbol{\varepsilon} \sim N(\boldsymbol{0}, \sigma^2 \boldsymbol{I}).$$

By Result 5.3, the NTGMM $\Longrightarrow \boldsymbol{y} \sim N(\boldsymbol{X\beta}, \sigma^2 \boldsymbol{I})$.

Mean of $\boldsymbol{y}$ determined by $\boldsymbol{\beta}$ through $\boldsymbol{X\beta}$.

Variance of $\boldsymbol{y}$ determined by σ^2.

$$\boldsymbol{y} = \boldsymbol{P_X y} + (\boldsymbol{I} - \boldsymbol{P_X})\boldsymbol{y} \quad \boldsymbol{P_X y} \in \mathcal{C}(\boldsymbol{X}),\ (\boldsymbol{I} - \boldsymbol{P_X})\boldsymbol{y} \in \mathcal{C}(\boldsymbol{X})^{\perp}.$$

We use $\hat{\boldsymbol{y}} = \boldsymbol{P_X y}$ to estimate $\boldsymbol{X}\boldsymbol{\beta} \quad (\boldsymbol{P_X y} = \boldsymbol{X}\hat{\boldsymbol{\beta}})$

We use $\hat{\boldsymbol{\varepsilon}} = (\boldsymbol{I} - \boldsymbol{P_X})\boldsymbol{y}$ to estimate σ^2. $\left(\hat{\sigma}^2 = \frac{\hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}}}{n - rank(\boldsymbol{X})}\right)$.

Also, recall that

$$\boldsymbol{y}'\boldsymbol{y} = \hat{\boldsymbol{y}}'\hat{\boldsymbol{y}} + \hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}}$$

$$\text{SSTO} = \text{SSR} + \text{SSE}.$$

Under the NTGMM, what can we say about the distribution of these sums of squares?

Lemma 5.1:

A $p \times p$ symmetric matrix $\boldsymbol{A}$ is idempotent with rank s iff $\exists$ a $p \times s$ matrix $\boldsymbol{G}$ with orthogonal columns such that

$$\boldsymbol{A} = \boldsymbol{G}\boldsymbol{G}'.$$

Proof of Lemma 5.1:

($\Longrightarrow$) By the Spectral Decomposition Theorem,

$$\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}' = \sum_{i=1}^{p} \lambda_i \boldsymbol{q}_i \boldsymbol{q}_i',$$

where

$$\boldsymbol{Q} = [\boldsymbol{q}_1, \ldots, \boldsymbol{q}_p], \quad \boldsymbol{Q}'\boldsymbol{Q} = \boldsymbol{I}, \quad \boldsymbol{\Lambda} = diag(\lambda_1, \ldots, \lambda_p).$$

Because $\boldsymbol{A}$ is idempotent,

$$\lambda_i \in \{0, 1\} \quad \forall\, i = 1, \ldots, p.$$

Because $rank(\boldsymbol{A}) = s$, $\exists$ exactly s of $\lambda_1, \ldots, \lambda_p$ equal to 1 and $p - s$ of $\lambda_1, \ldots, \lambda_p$ equal to 0. Let $i_1, \ldots, i_s$ be $\ni$

$$\lambda_{i_1} = \cdots = \lambda_{i_s} = 1.$$

Let $\boldsymbol{G} = [\boldsymbol{q}_{i_1}, \ldots, \boldsymbol{q}_{i_s}]$. Then

$$\boldsymbol{A} = \sum_{i=1}^{p} \lambda_i \boldsymbol{q}_i \boldsymbol{q}_i' = \sum_{j=1}^{s} \lambda_{i_j} \boldsymbol{q}_{i_j} \boldsymbol{q}_{i_j}'$$

$$= \sum_{j=1}^{s} \boldsymbol{q}_{i_j} \boldsymbol{q}_{i_j}' = \boldsymbol{G}\boldsymbol{G}' \quad \text{and} \quad \boldsymbol{G}'\boldsymbol{G} = \boldsymbol{I}.$$

$(\Longleftarrow)$ If $\boldsymbol{A} = \boldsymbol{G}\boldsymbol{G}'$, where $\boldsymbol{G}$ is a $p \times s$ matrix with orthonormal columns. Then

$$\begin{aligned} rank(\boldsymbol{G}\boldsymbol{G}') &= rank(\boldsymbol{G}') \\ &= rank(\boldsymbol{G}) \\ &= rank(\boldsymbol{G}'\boldsymbol{G}) \\ &= rank(\underset{s \times s}{\boldsymbol{I}}) \\ &= s. \end{aligned}$$

Thus, $rank(\boldsymbol{A}) = s$.

Furthermore, $\boldsymbol{A}' = (\boldsymbol{G}\boldsymbol{G}')' = \boldsymbol{G}\boldsymbol{G}' = \boldsymbol{A}$ and

$$\begin{aligned}\boldsymbol{A}\boldsymbol{A} &= (\boldsymbol{G}\boldsymbol{G}')(\boldsymbol{G}\boldsymbol{G}')\\ &= \boldsymbol{G}(\boldsymbol{G}'\boldsymbol{G})\boldsymbol{G}'\\ &= \boldsymbol{G}\boldsymbol{I}\boldsymbol{G}'\\ &= \boldsymbol{G}\boldsymbol{G}'\\ &= \boldsymbol{A}.\end{aligned}$$

$\therefore \boldsymbol{A}$ is also symmetric and idempotent. $\square$

Result 5.14:

Let $\boldsymbol{X} \sim N(\boldsymbol{\mu}, \underset{p \times p}{\boldsymbol{I}})$ and let $\underset{p \times p}{\boldsymbol{A}}$ be a symmetric matrix. Then

$$\boldsymbol{A} \text{ is idempotent with rank } s \Longrightarrow \boldsymbol{X}'\boldsymbol{A}\boldsymbol{X} \sim \chi^2_s(\boldsymbol{\mu}'\boldsymbol{A}\boldsymbol{\mu}/2).$$

Proof of Result 5.14:

By Lemma 5.1, $\exists\, \boldsymbol{G} \ni$

$$\boldsymbol{A} = \boldsymbol{G}\boldsymbol{G}' \quad \text{and} \quad \boldsymbol{G}'\boldsymbol{G} = \underset{s \times s}{\boldsymbol{I}}.$$

Then

$$\boldsymbol{G}'\boldsymbol{X} \sim N(\boldsymbol{G}'\boldsymbol{\mu}, \boldsymbol{G}'\boldsymbol{I}\boldsymbol{G} = \boldsymbol{G}'\boldsymbol{G} = \underset{s \times s}{\boldsymbol{I}}).$$

Thus, by Result 5.9,

$$(\boldsymbol{G}'\boldsymbol{X})'(\boldsymbol{G}'\boldsymbol{X}) \sim \chi^2_s((\boldsymbol{G}'\boldsymbol{\mu})'\boldsymbol{G}'\boldsymbol{\mu}/2).$$

$$(\boldsymbol{G}'\boldsymbol{X})'(\boldsymbol{G}'\boldsymbol{X}) = \boldsymbol{X}'\boldsymbol{G}\boldsymbol{G}'\boldsymbol{X} = \boldsymbol{X}'\boldsymbol{A}\boldsymbol{X}$$

$$(\boldsymbol{G}'\boldsymbol{\mu})'\boldsymbol{G}'\boldsymbol{\mu} = \boldsymbol{\mu}'\boldsymbol{G}\boldsymbol{G}'\boldsymbol{\mu} = \boldsymbol{\mu}'\boldsymbol{A}\boldsymbol{\mu}.$$

$\square$

Result 5.15:

Suppose $\boldsymbol{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, with $\underset{p \times p}{\boldsymbol{\Sigma}}$ of rank p. Suppose $\boldsymbol{A}$ is $p \times p$ and symmetric. Then

$$\boldsymbol{A\Sigma} \text{ is idempotent of rank } s \Longrightarrow \boldsymbol{X}'\boldsymbol{A}\boldsymbol{X} \sim \chi^2_s(\boldsymbol{\mu}'\boldsymbol{A}\boldsymbol{\mu}/2).$$

Proof of Result 5.15:

Let $\boldsymbol{W} = \boldsymbol{\Sigma}^{-1/2}\boldsymbol{X}$. Then

$$\boldsymbol{W} \sim N(\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\mu}, \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1/2} = \boldsymbol{I}).$$

Let $\boldsymbol{B} = \boldsymbol{\Sigma}^{1/2}\boldsymbol{A}\boldsymbol{\Sigma}^{1/2}$. Then $\boldsymbol{B}$ is symmetric by symmetry of $\boldsymbol{\Sigma}^{1/2}$ and $\boldsymbol{A}$. Furthermore,

$$\begin{aligned} rank(\boldsymbol{B}) &= rank(\boldsymbol{\Sigma}^{1/2}\boldsymbol{A}\boldsymbol{\Sigma}^{1/2}) = rank(\boldsymbol{A}) \\ &= rank(\boldsymbol{A}\boldsymbol{\Sigma}) = s. \end{aligned}$$

$\because$ $\boldsymbol{\Sigma}^{1/2}$ and $\boldsymbol{\Sigma}$ are full-rank.

Finally, note that $\boldsymbol{B}$ is idempotent:

$$\begin{aligned}
\boldsymbol{A\Sigma A\Sigma} = \boldsymbol{A\Sigma} &\iff \boldsymbol{\Sigma}^{1/2}\boldsymbol{A\Sigma A\Sigma} = \boldsymbol{\Sigma}^{1/2}\boldsymbol{A\Sigma} \\
&\iff \boldsymbol{\Sigma}^{1/2}\boldsymbol{A\Sigma A\Sigma\Sigma}^{-1/2} = \boldsymbol{\Sigma}^{1/2}\boldsymbol{A\Sigma\Sigma}^{-1/2} \\
&\iff \boldsymbol{\Sigma}^{1/2}\boldsymbol{A\Sigma A\Sigma}^{1/2} = \boldsymbol{\Sigma}^{1/2}\boldsymbol{A\Sigma}^{1/2} \\
&\iff \boldsymbol{\Sigma}^{1/2}\boldsymbol{A\Sigma}^{1/2}\boldsymbol{\Sigma}^{1/2}\boldsymbol{A\Sigma}^{1/2} = \boldsymbol{\Sigma}^{1/2}\boldsymbol{A\Sigma}^{1/2} \\
&\iff \boldsymbol{BB} = \boldsymbol{B}.
\end{aligned}$$

Thus, by Result 5.14,

$$\boldsymbol{W}'\boldsymbol{B}\boldsymbol{W} \sim \chi^2_s((\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\mu})'\boldsymbol{B}(\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\mu})/2).$$

Now note

$$\begin{aligned} \boldsymbol{W}'\boldsymbol{B}\boldsymbol{W} &= \boldsymbol{X}'\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Sigma}^{1/2}\boldsymbol{A}\boldsymbol{\Sigma}^{1/2}\boldsymbol{\Sigma}^{-1/2}\boldsymbol{X} \\ &= \boldsymbol{X}'\boldsymbol{A}\boldsymbol{X} \quad \text{and likewise} \\ (\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\mu})'\boldsymbol{B}(\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\mu}) &= \boldsymbol{\mu}'\boldsymbol{A}\boldsymbol{\mu}. \end{aligned}$$

$\therefore \boldsymbol{X}'\boldsymbol{A}\boldsymbol{X} \sim \chi^2_s(\boldsymbol{\mu}'\boldsymbol{A}\boldsymbol{\mu}/2)$. $\square$

Find the distribution of SSE.

$$\begin{aligned} \text{SSE} &= \hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}} = \boldsymbol{y}'(\boldsymbol{I} - \boldsymbol{P_X})\boldsymbol{y} \\ &= \sigma^2 \boldsymbol{y}' \left(\frac{\boldsymbol{I} - \boldsymbol{P_X}}{\sigma^2} \right) \boldsymbol{y}. \end{aligned}$$

Let

$$\boldsymbol{A} = \frac{\boldsymbol{I} - \boldsymbol{P_X}}{\sigma^2} \quad \text{and} \quad \boldsymbol{\Sigma} = \sigma^2 \boldsymbol{I} = \text{Var}(\boldsymbol{y}).$$

Then

$$\boldsymbol{A\Sigma} = \frac{\boldsymbol{I} - \boldsymbol{P_X}}{\sigma^2} \sigma^2 \boldsymbol{I} = \boldsymbol{I} - \boldsymbol{P_X}.$$

Thus, $\boldsymbol{A\Sigma}$ is idempotent and $rank(\boldsymbol{A\Sigma}) = n - rank(\boldsymbol{X})$.

By Result 5.15,

$$\boldsymbol{y}'\left(\frac{\boldsymbol{I}-\boldsymbol{P_X}}{\sigma^2}\right)\boldsymbol{y} \sim \chi^2_{n-rank(\boldsymbol{X})}\left(\frac{1}{2}\boldsymbol{\beta}'\boldsymbol{X}'\left(\frac{\boldsymbol{I}-\boldsymbol{P_X}}{\sigma^2}\right)\boldsymbol{X}\boldsymbol{\beta}\right).$$

$\because (\boldsymbol{I}-\boldsymbol{P_X})\boldsymbol{X} = \boldsymbol{X} - \boldsymbol{P_X}\boldsymbol{X} = \boldsymbol{X} - \boldsymbol{X} = \boldsymbol{0}$, we have

$$\boldsymbol{y}'\left(\frac{\boldsymbol{I}-\boldsymbol{P_X}}{\sigma^2}\right)\boldsymbol{y} \sim \chi^2_{n-rank(\boldsymbol{X})}.$$

Thus,

$$\text{SSE} \sim \sigma^2\chi^2_{n-rank(\boldsymbol{X})},$$

which is a scaled Chi-Square distribution.

Similarly,

$$\frac{\text{SSR}}{\sigma^2} \sim \chi^2_{rank(\boldsymbol{X})}\left(\frac{1}{2}\boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{\beta}/\sigma^2\right).$$

Result 5.16:

Suppose $\boldsymbol{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\boldsymbol{A}$ is symmetric with rank s. Then

$$\boldsymbol{B\Sigma A} = \boldsymbol{0} \Longrightarrow \boldsymbol{BX} \text{ and } \boldsymbol{X}'\boldsymbol{AX} \text{ are independent.}$$

By the SDT, we have

$$\underset{p\times p}{\boldsymbol{A}} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}',$$

where $\boldsymbol{Q}$ is square with orthonormal columns $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_p$ and $\boldsymbol{\Lambda} = diag(\lambda_1, \ldots, \lambda_p)$ with exactly s of $\lambda_1, \ldots, \lambda_p$ not equal to zero. Because

$$\boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}' = \sum_{i=1}^{p} \lambda_i \boldsymbol{q}_i \boldsymbol{q}_i',$$

we can without loss of generality (WLOG) assume

$$\lambda_1, \ldots, \lambda_s \neq 0.$$

Thus,

$$\boldsymbol{A} = \sum_{i=1}^{s} \lambda_i \boldsymbol{q}_i \boldsymbol{q}_i' = \boldsymbol{Q}_1 \boldsymbol{\Lambda}_1 \boldsymbol{Q}_1',$$

where

$$\boldsymbol{Q}_1 = [\boldsymbol{q}_1, \ldots, \boldsymbol{q}_s], \; \boldsymbol{Q}_1'\boldsymbol{Q}_1 = \underset{s \times s}{\boldsymbol{I}}$$

$$\boldsymbol{\Lambda}_1 = diag(\lambda_1, \ldots, \lambda_s) \quad \text{and} \quad \boldsymbol{\Lambda}_1^{-1} = diag\left(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_s}\right).$$

Now consider $\begin{bmatrix} \boldsymbol{BX} \\ \boldsymbol{Q}_1'\boldsymbol{X} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{Q}_1' \end{bmatrix} \boldsymbol{X}$. Then

$$\begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{Q}_1' \end{bmatrix} \boldsymbol{X} \sim N\left(\begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{Q}_1' \end{bmatrix} \boldsymbol{\mu}, \boldsymbol{V} \right),$$

where

$$\boldsymbol{V} = \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{Q}_1' \end{bmatrix} \boldsymbol{\Sigma} \begin{bmatrix} \boldsymbol{B}' & \boldsymbol{Q}_1 \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{B\Sigma B}' & \boldsymbol{B\Sigma Q}_1 \\ \boldsymbol{Q}_1'\boldsymbol{\Sigma B}' & \boldsymbol{Q}_1'\boldsymbol{\Sigma Q}_1 \end{bmatrix}.$$

$$
\begin{aligned}
\boldsymbol{B\Sigma A} = \mathbf{0} &\Longrightarrow \boldsymbol{B\Sigma Q}_1 \boldsymbol{\Lambda}_1 \boldsymbol{Q}_1' = \mathbf{0} \\
&\Longrightarrow \boldsymbol{B\Sigma Q}_1 \boldsymbol{\Lambda}_1 \boldsymbol{Q}_1' \boldsymbol{Q}_1 = \mathbf{0}\boldsymbol{Q}_1 \\
&\Longrightarrow \boldsymbol{B\Sigma Q}_1 \boldsymbol{\Lambda}_1 = \mathbf{0} \\
&\Longrightarrow \boldsymbol{B\Sigma Q}_1 \boldsymbol{\Lambda}_1 \boldsymbol{\Lambda}_1^{-1} = \mathbf{0}\boldsymbol{\Lambda}_1^{-1} \\
&\Longrightarrow \boldsymbol{B\Sigma Q}_1 = \mathbf{0} \\
&\Longrightarrow \boldsymbol{BX} \text{ and } \boldsymbol{Q}_1'\boldsymbol{X} \text{ independent by Result 5.4.}
\end{aligned}
$$

Now $\boldsymbol{BX}$ and $\boldsymbol{Q}_1'\boldsymbol{X}$ are independent,

$$\Longrightarrow \boldsymbol{BX} \text{ and } (\boldsymbol{Q}_1'\boldsymbol{X})'\boldsymbol{\Lambda}_1\boldsymbol{Q}_1'\boldsymbol{X} \text{ independent}$$

$$\Longrightarrow \boldsymbol{BX} \text{ and } \boldsymbol{X}'\boldsymbol{Q}_1\boldsymbol{\Lambda}_1\boldsymbol{Q}_1'\boldsymbol{X} \text{ independent}$$

$$\Longrightarrow \boldsymbol{BX} \text{ and } \boldsymbol{X}'\boldsymbol{AX} \text{ independent.}$$

□

Corollary 5.4:

Suppose $\boldsymbol{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $\boldsymbol{A}$ symmetric with rank r, $\boldsymbol{B}$ symmetric with rank s. Then

$$\boldsymbol{B\Sigma A} = \boldsymbol{0} \Longrightarrow \boldsymbol{X}'\boldsymbol{AX} \text{ and } \boldsymbol{X}'\boldsymbol{BX} \text{ are independent.}$$

Proof:

HW problem. $\square$

Find the distribution of

$$\frac{\text{SSR}/r}{\text{SSE}/(n-r)}, \quad \text{where} \quad r = rank(\boldsymbol{X}).$$

We have seen that

$$\frac{\text{SSR}}{\sigma^2} \sim \chi_r^2\left(\frac{1}{2}\boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{\beta}/\sigma^2\right)$$

and

$$\frac{\text{SSE}}{\sigma^2} \sim \chi_{n-r}^2.$$

Thus,

$$\frac{\text{SSR}/r}{\text{SSE}/(n-r)} = \frac{\frac{\text{SSR}}{\sigma^2}/r}{\frac{\text{SSE}}{\sigma^2}/(n-r)}$$

will have the distribution $F_{r,n-r}\left(\frac{1}{2}\boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{\beta}/\sigma^2\right)$ if SSR and SSE independent.

$$\text{SSR} = \boldsymbol{y}'\boldsymbol{P_X}\boldsymbol{y}, \qquad \text{SSE} = \boldsymbol{y}'(\boldsymbol{I} - \boldsymbol{P_X})\boldsymbol{y}$$

$$\begin{aligned}(\boldsymbol{I} - \boldsymbol{P_X})(\sigma^2\boldsymbol{I})\boldsymbol{P_X} &= \sigma^2(\boldsymbol{I} - \boldsymbol{P_X})\boldsymbol{P_X} \\ &= \sigma^2(\boldsymbol{P_X} - \boldsymbol{P_X}\boldsymbol{P_X}) \\ &= \sigma^2(\boldsymbol{P_X} - \boldsymbol{P_X}) \\ &= \boldsymbol{0}.\end{aligned}$$

By Corollary 5.4, SSR and SSE are independent.