Eigenvalues, Eigenvectors, and Matrix Decompositions

$\underset{n\times n}{\boldsymbol{A}} \Rightarrow q(\lambda) = |\boldsymbol{A} - \lambda\boldsymbol{I}|$ is a polynomial of degree n.

For example,

$$\underset{2\times 2}{\boldsymbol{A}} \Rightarrow |\boldsymbol{A} - \lambda\boldsymbol{I}| = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix}$$

$$= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21}$$

$$= \lambda^2 + (-a_{11} - a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}.$$

The definition of $|\boldsymbol{A} - \lambda \boldsymbol{I}|$ implies that the coefficient on λ^n is $(-1)^n$.

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} - \lambda & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} - \lambda \end{vmatrix}$$

The n (not necessarily distinct) roots of $q(\lambda)$ (i.e., solutions to $q(\lambda) = 0$) are called the eigenvalues of $\boldsymbol{A}$.

The Fundamental Theorem of Algebra guarantees n roots, and the Polynomial Factorization Theorem implies that

$$q(\lambda) = |\boldsymbol{A} - \lambda \boldsymbol{I}| = \prod_{i=1}^{n} (\lambda_i - \lambda)$$

for the eigenvalues $\lambda_1, \ldots, \lambda_n$.

Example:

Suppose $\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, then

$$\begin{aligned} |\boldsymbol{A} - \lambda \boldsymbol{I}| &= \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} \\ &= (2-\lambda)(2-\lambda) - 1 \\ &= \lambda^2 - 4\lambda + 3 \\ &= (3-\lambda)(1-\lambda). \end{aligned}$$

Eigenvalues are

$\lambda_1 = 3, \lambda_2 = 1 \because (3-\lambda)(1-\lambda) = 0 \Longleftrightarrow \lambda = 3$ or $\lambda = 1$.

Example:

Suppose $\boldsymbol{A} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, then

$$\begin{aligned} |\boldsymbol{A} - \lambda \boldsymbol{I}| &= \begin{vmatrix} 3 - \lambda & 0 \\ 0 & 3 - \lambda \end{vmatrix} \\ &= (3 - \lambda)(3 - \lambda) \\ &= \lambda^2 - 6\lambda + 9. \end{aligned}$$

Eigenvalues are $\lambda_1 = 3, \lambda_2 = 3 \because (3 - \lambda)(3 - \lambda) = 0 \iff \lambda = 3.$

In the previous example, the eigenvalue 3 is said to have algebraic multiplicity 2.

The algebraic multiplicity of eigenvalue λ^* is

$$\sum_{i=1}^{n} \mathbb{1}[\lambda_i = \lambda^*].$$

Note that the roots of $q(\lambda)$ are not necessarily real numbers.

For example,

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow \boldsymbol{A} - \lambda\boldsymbol{I} = \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}$$

$$\Rightarrow |\boldsymbol{A} - \lambda\boldsymbol{I}| = \lambda^2 + 1$$

$$= (\sqrt{-1} - \lambda)(-\sqrt{-1} - \lambda)$$

$$\therefore \lambda_1 = \sqrt{-1}, \lambda_2 = -\sqrt{-1}.$$

If $\underset{n \times n}{\boldsymbol{A}}$ is symmetric, then all roots of $q(\lambda) = |\boldsymbol{A} - \lambda \boldsymbol{I}|$ are real numbers.

λ is an eigenvalue of $\boldsymbol{A}$

$$\iff |\boldsymbol{A} - \lambda\boldsymbol{I}| = 0$$

$$\iff \boldsymbol{A} - \lambda\boldsymbol{I} \text{ is singular}$$

$$\iff rank(\boldsymbol{A} - \lambda\boldsymbol{I}) < n$$

$$\iff \exists\, \boldsymbol{x} \neq \boldsymbol{0} \ni (\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$$

$$\iff \exists\, \boldsymbol{x} \neq \boldsymbol{0} \ni \boldsymbol{A}\boldsymbol{x} - \lambda\boldsymbol{x} = \boldsymbol{0}$$

$$\iff \exists\, \boldsymbol{x} \neq \boldsymbol{0} \ni \boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}.$$

Such a vector $\boldsymbol{x} \neq \mathbf{0}$ satisfying $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$ is called an eigenvector of $\underset{n\times n}{\boldsymbol{A}}$ corresponding to eigenvalue λ.

Note that if $\boldsymbol{x}$ is an eigenvector corresponding to eigenvalue λ, then $c\boldsymbol{x}$ is also an eigenvector $\forall\, c \in \mathbb{R}$.

$\because \boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x} \Rightarrow c\boldsymbol{A}\boldsymbol{x} = c\lambda\boldsymbol{x} \Rightarrow \boldsymbol{A}(c\boldsymbol{x}) = \lambda(c\boldsymbol{x})$.

It is customary to take eigenvalues to be unit norm, i.e., we take

$$c = \frac{1}{\|\boldsymbol{x}\|}$$

so that the eigenvector $c\boldsymbol{x}$ satisfies

$$\|c\boldsymbol{x}\| = 1.$$

If λ is an eigenvalue of $\underset{n \times n}{\boldsymbol{A}}$, the set of vectors satisfying $\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$ is called the <u>eigenspace</u> of the eigenvalue λ.

Note that the eigenspace

$$\{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}\} = \{\boldsymbol{x} : (\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}\} = \mathcal{N}(\boldsymbol{A} - \lambda\boldsymbol{I}).$$

The dimension of the eigenspace of λ is known as the geometric multiplicity of λ.

For symmetric matrices, it turns out that algebraic multiplicity of an eigenvalue is the same as the geometric multiplicity.

Show that eigenvectors corresponding to distinct eigenvalues of a symmetric matrix $\underset{n\times n}{\boldsymbol{A}}$ are orthogonal.

Proof:

Suppose $\boldsymbol{A}\boldsymbol{x}_1 = \lambda_1 \boldsymbol{x}_1, \boldsymbol{A}\boldsymbol{x}_2 = \lambda_2 \boldsymbol{x}_2$ and $\lambda_1 \neq \lambda_2$.

Then

$$\begin{aligned} \lambda_1 \boldsymbol{x}_1' \boldsymbol{x}_2 &= (\lambda_1 \boldsymbol{x}_1)' \boldsymbol{x}_2 = (\boldsymbol{A}\boldsymbol{x}_1)' \boldsymbol{x}_2 = \boldsymbol{x}_1' \boldsymbol{A}' \boldsymbol{x}_2 = \boldsymbol{x}_1' \boldsymbol{A} \boldsymbol{x}_2 \\ &= \boldsymbol{x}_1' (\boldsymbol{A}\boldsymbol{x}_2) = \boldsymbol{x}_1' (\lambda_2 \boldsymbol{x}_2) = \lambda_2 \boldsymbol{x}_1' \boldsymbol{x}_2. \end{aligned}$$

Now

$$\lambda_1 \boldsymbol{x}_1' \boldsymbol{x}_2 = \lambda_2 \boldsymbol{x}_1' \boldsymbol{x}_2 \Rightarrow (\lambda_1 - \lambda_2) \boldsymbol{x}_1' \boldsymbol{x}_2 = 0 \Rightarrow \boldsymbol{x}_1' \boldsymbol{x}_2 = 0.$$

□

Fact V7:

If $\mathcal{S} \subseteq \mathbb{R}^n$ is a vector space of dimension $k > 0$, then there exist a set of mutually orthonormal vectors that forms a basis for $\mathcal{S}$.

Proof: Homework problem.

It follows that for each distinct eigenvalue λ of a matrix $\underset{n\times n}{\boldsymbol{A}}$ satisfying $\boldsymbol{A} = \boldsymbol{A}'$, there exists a set of orthonormal eigenvectors with $dim(\mathcal{N}(\boldsymbol{A} - \lambda\boldsymbol{I}))$ elements.

These orthonormal eigenvectors form a basis for the eigenspace of λ.

Furthermore, the collection of all such eigenvectors corresponding to the eigenvalues of $\boldsymbol{A}$ is a mutually orthonormal set of n vectors.

Let $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ denote these n orthonormal eigenvectors corresponding to eigenvalues $\lambda_1, \ldots, \lambda_n$ respectively.

Let $\boldsymbol{Q} = [\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n]$.

Let $\boldsymbol{\Lambda} = diag(\lambda_1, \ldots, \lambda_n)$.

$\because \boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ are mutually orthonormal, $\boldsymbol{Q}'\boldsymbol{Q} = \boldsymbol{I}$.

Moreover, because $\boldsymbol{Q}$ is $n \times n$, it follows that $\boldsymbol{Q}'$ is $\boldsymbol{Q}^{-1}$.

Thus,

$$\boldsymbol{Q}\boldsymbol{Q}' = \boldsymbol{Q}'\boldsymbol{Q} = \boldsymbol{I}.$$

Note that

$$
\begin{aligned}
& \boldsymbol{A}\boldsymbol{q}_i = \lambda_i \boldsymbol{q}_i \;\forall\; i = 1, \ldots, n \\
& \iff \boldsymbol{A}\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{\Lambda} \\
& \iff \boldsymbol{A}\boldsymbol{Q}\boldsymbol{Q}' = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}' \\
& \iff \boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}' = \sum_{i=1}^{n} \lambda_i \boldsymbol{q}_i \boldsymbol{q}_i'.
\end{aligned}
$$

This result is known as the Spectral Decomposition Theorem:

If $\underset{n\times n}{\boldsymbol{A}}$ is a symmetric matrix, then $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}'$, where $\boldsymbol{Q}$ is an orthogonal matrix and $\boldsymbol{\Lambda}$ is a diagonal matrix.

Result A.19:

Let $\underset{n \times n}{\boldsymbol{A}}$ be a symmetric matrix. Then $rank(\boldsymbol{A})$ is equal to the number of nonzero eigenvalues of $\boldsymbol{A}$.

Proof: Homework problem.

Suppose $\underset{n\times n}{A}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$. Then

(a) $trace(A) = \sum_{i=1}^{n} \lambda_i$, and

(b) $|A| = \prod_{i=1}^{n} \lambda_i$.

Proof: Homework problem.

Quadratic Forms

Suppose $\underset{n \times n}{\boldsymbol{A}} = [a_{ij}]$ and $\boldsymbol{x} = [x_1, \ldots, x_n]'$.

$\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j a_{ij}$ is known as a quadratic form.

Nonnegative Definite and Positive Definite Matrices

A symmetric matrix $\underset{n\times n}{\boldsymbol{A}}$ is nonnegative definite (NND) if and only if

$$\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} \geq 0 \ \forall \ \boldsymbol{x} \in \mathbb{R}^n.$$

A symmetric matrix $\underset{n\times n}{\boldsymbol{A}}$ is positive definite (PD) if and only if

$$\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} > 0 \ \forall \ \boldsymbol{x} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}.$$

Suppose $\underset{n\times n}{\boldsymbol{A}}$ is a symmetric matrix.

Prove that $\boldsymbol{A}$ is NND if and only if all eigenvalues of $\boldsymbol{A}$ are nonnegative.

Prove that $\boldsymbol{A}$ is PD if and only if all eigenvalues of $\boldsymbol{A}$ are positive.

Proof:

$(\Longrightarrow)$ Suppose $\boldsymbol{A}$ is a symmetric NND matrix. Then $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}'$ and $\boldsymbol{q}_i'\boldsymbol{A}\boldsymbol{q}_i \geq 0 \; \forall \; i = 1, \ldots, n$.

Now note that $\boldsymbol{Q}'\boldsymbol{q}_i = \boldsymbol{e}_i$ (the ith column of $\underset{n \times n}{\boldsymbol{I}}$). Thus,

$$\lambda_i = \boldsymbol{e}_i'\boldsymbol{\Lambda}\boldsymbol{e}_i = \boldsymbol{q}_i'\boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}'\boldsymbol{q}_i = \boldsymbol{q}_i'\boldsymbol{A}\boldsymbol{q}_i \geq 0.$$

($\Longleftarrow$) Suppose all eigenvalues of $\boldsymbol{A}, \lambda_1, \ldots, \lambda_n$, are nonnegative.

Let $\boldsymbol{x} \in \mathbb{R}^n$ be arbitrary.

Let $\boldsymbol{y} = \boldsymbol{Q}\boldsymbol{x}$. Then

$$\begin{aligned}\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} &= \boldsymbol{x}'\boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}'\boldsymbol{x} \\ &= \boldsymbol{y}'\boldsymbol{\Lambda}\boldsymbol{y} \\ &= \sum_{i=1}^{n} \lambda_i y_i^2 \geq 0.\end{aligned}$$

$\square$

Suppose $\underset{n\times n}{\boldsymbol{A}}$ is a symmetric NND matrix.

Show there exists a symmetric matrix $\boldsymbol{B}$ such that $\boldsymbol{BB} = \boldsymbol{A}$.

Such a matrix is called the symmetric square root of $\boldsymbol{A}$ and is denoted by $\boldsymbol{A}^{1/2}$.

Proof:

Suppose $\underset{n\times n}{\boldsymbol{A}}$ is a symmetric NND matrix.

By the Spectral Decomposition Theorem, $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}'$ where $\underset{n\times n}{\boldsymbol{Q}}$ is orthogonal and $\boldsymbol{\Lambda}$ is diagonal with eigenvalues of $\boldsymbol{A}$ on the diagonal.

Because $\underset{n\times n}{\boldsymbol{A}}$ is NND, eigenvalues of $\underset{n\times n}{\boldsymbol{A}}$ are nonnegative. Define

$$\boldsymbol{\Lambda}^{1/2} = diag(\lambda_1^{1/2}, \ldots, \lambda_n^{1/2}),$$

where

$$\boldsymbol{\Lambda} = diag(\lambda_1, \ldots, \lambda_n).$$

Now define $\boldsymbol{B} = \boldsymbol{Q}\boldsymbol{\Lambda}^{1/2}\boldsymbol{Q}'$. Note

$$\begin{aligned}\boldsymbol{B}' &= (\boldsymbol{Q}\boldsymbol{\Lambda}^{1/2}\boldsymbol{Q}')' \\ &= (\boldsymbol{Q}')'(\boldsymbol{\Lambda}^{1/2})'\boldsymbol{Q}' \\ &= \boldsymbol{Q}\boldsymbol{\Lambda}^{1/2}\boldsymbol{Q}' = \boldsymbol{B}.\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{B}\boldsymbol{B} &= \boldsymbol{Q}\boldsymbol{\Lambda}^{1/2}\boldsymbol{Q}'\boldsymbol{Q}\boldsymbol{\Lambda}^{1/2}\boldsymbol{Q}' \\ &= \boldsymbol{Q}\boldsymbol{\Lambda}^{1/2}\boldsymbol{\Lambda}^{1/2}\boldsymbol{Q}' \\ &= \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}' = \boldsymbol{A}.\end{aligned}$$

Note that if $\boldsymbol{A}$ is PD, $\boldsymbol{A}^{1/2} = \boldsymbol{Q}\boldsymbol{\Lambda}^{1/2}\boldsymbol{Q}'$ is nonsingular

$$\because \boldsymbol{A}^{-1/2} = \boldsymbol{Q}\left[diag(1/\sqrt{\lambda_1}, \ldots, 1/\sqrt{\lambda_1})\right]\boldsymbol{Q}' = \boldsymbol{Q}\boldsymbol{\Lambda}^{-1/2}\boldsymbol{Q}'$$

is its inverse:

$$\boldsymbol{A}^{1/2}\boldsymbol{A}^{-1/2} = \boldsymbol{Q}\boldsymbol{\Lambda}^{1/2}\boldsymbol{Q}'\boldsymbol{Q}\boldsymbol{\Lambda}^{-1/2}\boldsymbol{Q}' = \boldsymbol{Q}\boldsymbol{\Lambda}^{1/2}\boldsymbol{\Lambda}^{-1/2}\boldsymbol{Q}' = \boldsymbol{Q}\boldsymbol{Q}' = \boldsymbol{I}.$$

□

Result A.20 (Cholesky Decomposition):

Suppose $\underset{n\times n}{\boldsymbol{A}}$ is a symmetric matrix. $\underset{n\times n}{\boldsymbol{A}}$ is PD iff there exists a nonsingular lower triangular matrix $\boldsymbol{L}$ such that

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{L}'.$$

Proof of Result A.20:

$(\Longleftarrow)$ $\forall\, \boldsymbol{x} \neq \mathbf{0}, \boldsymbol{L}'\boldsymbol{x} \neq \mathbf{0} \because \boldsymbol{L}'$ is nonsingular and thus full column rank.

Thus, $\forall\, \boldsymbol{x} \neq \mathbf{0}$,

$$\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}'\boldsymbol{L}\boldsymbol{L}'\boldsymbol{x} = (\boldsymbol{L}'\boldsymbol{x})'(\boldsymbol{L}'\boldsymbol{x}) > 0.$$

$(\Longrightarrow)$ If $n = 1, \underset{1\times 1}{\boldsymbol{A}} = [a]$. Moreover,

$$\begin{aligned} \underset{1\times 1}{\boldsymbol{A}} \text{ is PD } &\Rightarrow x'ax > 0 \;\forall\; x \neq 0 \\ &\Rightarrow ax^2 > 0 \;\forall\; x \neq 0 \\ &\Rightarrow a > 0. \end{aligned}$$

If we take $\boldsymbol{L} = [\sqrt{a}]$, then $\boldsymbol{L}\boldsymbol{L}' = \boldsymbol{A}$, $\boldsymbol{L}$ is a nonsingular lower triangular matrix, and the result holds.

Now suppose the result holds for $n = 1, \ldots, k$ for some integer $k \geq 1$.

Suppose $\boldsymbol{A}$ is any $(k+1) \times (k+1)$ matrix. Partition $\boldsymbol{A}$ as

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_k & \boldsymbol{a}_k \\ \boldsymbol{a}_k' & a \end{bmatrix}$$

$\boldsymbol{A}$ is PD $\Rightarrow \boldsymbol{A}_k$ is PD $\because$

$$\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} > 0 \ \forall \ \boldsymbol{x} \neq \mathbf{0}$$

$$\Rightarrow \boldsymbol{x}'\boldsymbol{A}\boldsymbol{x} > 0 \ \forall \ \boldsymbol{x} \neq \mathbf{0} \text{ with the last component } 0$$

$$\Rightarrow [\boldsymbol{y}' \ 0]\boldsymbol{A} \begin{bmatrix} \boldsymbol{y} \\ 0 \end{bmatrix} > 0 \ \forall \ \boldsymbol{y} \neq \mathbf{0}$$

$$\Rightarrow \boldsymbol{y}'\boldsymbol{A}_k\boldsymbol{y} > 0 \ \forall \ \boldsymbol{y} \neq \mathbf{0}.$$

$\therefore \exists\, \boldsymbol{L}_k$ nonsingular and lower triangular $\ni \boldsymbol{A}_k = \boldsymbol{L}_k \boldsymbol{L}_k'$.

Now let $\boldsymbol{l}_k = \boldsymbol{L}_k^{-1} \boldsymbol{a}_k$.

$\because \boldsymbol{A}$ is PD,

$$\begin{bmatrix} \boldsymbol{l}_k' \boldsymbol{L}_k^{-1} & -1 \end{bmatrix} \begin{bmatrix} \boldsymbol{A}_k & \boldsymbol{a}_k \\ \boldsymbol{a}_k' & a \end{bmatrix} \begin{bmatrix} \boldsymbol{L}_k'^{-1} \boldsymbol{l}_k \\ -1 \end{bmatrix} > 0 \Rightarrow$$

$$\Rightarrow \boldsymbol{l}_k' \boldsymbol{L}_k^{-1} \boldsymbol{A}_k \boldsymbol{L}_k'^{-1} \boldsymbol{l}_k - 2\boldsymbol{l}_k' \boldsymbol{L}_k^{-1} \boldsymbol{a}_k + a > 0$$

$$\Rightarrow \boldsymbol{l}_k' \boldsymbol{L}_k^{-1} \boldsymbol{L}_k \boldsymbol{L}_k' \boldsymbol{L}_k'^{-1} \boldsymbol{l}_k - 2\boldsymbol{l}_k' \boldsymbol{l}_k + a > 0$$

$$\Rightarrow -\boldsymbol{l}_k' \boldsymbol{l}_k + a > 0.$$

Now let $l = \sqrt{-\boldsymbol{l}_k'\boldsymbol{l}_k + a}$.

Then

$$\boldsymbol{L} = \begin{bmatrix} \boldsymbol{L}_k & 0 \\ \boldsymbol{l}_k' & l \end{bmatrix}$$

is lower triangular and nonsingular ($\because l > 0 \Rightarrow L$ is full rank) and

$$
\begin{aligned}
\boldsymbol{L}\boldsymbol{L}' &= \begin{bmatrix} \boldsymbol{L}_k & 0 \\ \boldsymbol{l}_k' & l \end{bmatrix} \begin{bmatrix} \boldsymbol{L}_k' & \boldsymbol{l}_k \\ 0 & l \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{L}_k\boldsymbol{L}_k' & \boldsymbol{L}_k\boldsymbol{l}_k \\ \boldsymbol{l}_k'\boldsymbol{L}_k' & \boldsymbol{l}_k'\boldsymbol{l}_k + l^2 \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{A}_k & \boldsymbol{L}_k\boldsymbol{L}_k^{-1}\boldsymbol{a}_k \\ \boldsymbol{a}_k'\boldsymbol{L}_k'^{-1}\boldsymbol{L}_k' & \boldsymbol{l}_k'\boldsymbol{l}_k + a - \boldsymbol{l}_k'\boldsymbol{l}_k \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{A}_k & \boldsymbol{a}_k \\ \boldsymbol{a}_k' & a \end{bmatrix} \\
&= \boldsymbol{A}.
\end{aligned}
$$

□