Determinants
The determinant of a square matrix $A$ is denoted $\det(A)$ or $|A|$. 
Definition of Determinant

The determinant of \( A = [a_{ij}] \) is defined as

\[
|A| = \sum_{k=1}^{n!} (-1)^{m(p_k)} \prod_{i=1}^{n} a_{ip_k(i)},
\]

where \( p_1, \ldots, p_n! \) are the \( n! \) distinct bijections (one-to-one and onto functions) from \( \{1, \ldots, n\} \) to \( \{1, \ldots, n\} \) and

\[
m(p_k) = \sum_{i=1}^{n-1} \sum_{i^* = i+1}^{n} \mathbb{1}[p_k(i) > p_k(i^*)].
\]
Example: $n = 2$

Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$ 

Then $n! = 2$,

$$p_1(1) = 1 \quad p_1(2) = 2 \quad m(p_1) = 0$$
$$p_2(1) = 2 \quad p_2(2) = 1 \quad m(p_2) = 1,$$

and

$$|A| = \sum_{k=1}^{2} (-1)^{m(p_k)} \prod_{i=1}^{2} a_{ip_k(i)}$$
$$= a_{11}a_{22} - a_{12}a_{21}.$$
Example: $n = 3$

Find an expression for the determinant of

$$A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}.$$
Example: \( n = 1 \)

Suppose \( A = [a_{11}] \). Then \( n! = 1 \),

\[
p_1(1) = 1 \quad m(p_1) = 0,
\]

and

\[
|A| = \sum_{k=1}^{1} (-1)^{m(p_k)} \prod_{i=1}^{1} a_{ip_k(i)} = a_{11}.
\]
Verbal Description of Determinant Computation

1. Form all possible products of $n$ matrix elements that contain exactly one element from each row and exactly one element from each column.

2. For each product, count the number of pairs of elements for which the “lower” element is to the “left” of the “higher” element. (For example, $a_{31}$ is “lower” than $a_{12}$ because $a_{31}$ is in the third row while $a_{12}$ is in the first row. Also, $a_{31}$ is to the “left” of $a_{12}$ because $a_{31}$ is in the first column while $a_{12}$ is in the second column.) If the number of such pairs is odd, multiply the product by $-1$. If the number of pairs is even, multiply the product by $1$.

3. Add the signed products determined in steps 1 and 2.
Prove that if \( A \) is an \( n \times n \) matrix and \( c \in \mathbb{R} \), then

\[
|cA| = c^n |A|.
\]
Prove that if $A_{n \times n}$ is lower or upper triangular, then

$$|A| = \prod_{i=1}^{n} a_{ii}.$$  

(Note that this result implies $|A| = \prod_{i=1}^{n} a_{ii}$ for a diagonal matrix.)
Suppose \( A \) is an \( n \times n \) matrix. Prove that \( |A| = |A'| \).
Proof:

Let \( q_k = p_k^{-1} \); i.e., \( q_k(j) = i \iff p_k(i) = j \).

Note that

\[
i < i^* \quad \text{and} \quad p_k(i) = j > p_k(i^*) = j^* \iff j^* < j, \quad q_k(j^*) = i^* > q_k(j) = i.
\]

Thus, \( m(p_k) = m(q_k) \).
Proof:

It follows that

\[ |A| = \sum_{k=1}^{n!} (-1)^{m(p_k)} \prod_{i=1}^{n} a_{ip_k(i)} \]

\[ = \sum_{k=1}^{n!} (-1)^{m(p_k)} \prod_{j=1}^{n} a_{q_k(j)j} \]

\[ = \sum_{k=1}^{n!} (-1)^{m(q_k)} \prod_{j=1}^{n} a_{q_k(j)j} \]

\[ = |A'|. \]
Find an expression for

\[
\begin{vmatrix}
I_{n \times n} & 0_{n \times m} \\
B_{m \times n} & C_{m \times m}
\end{vmatrix} =
\begin{vmatrix}
I_{n \times n} & 0_{n \times m} \\
B_{m \times n} & C_{m \times m}
\end{vmatrix}.
\]
Result A.18 (b):

If $A$ and $B$ are each $n \times n$ matrices, then

$$|AB| = |A||B|.$$ 

(We will not go through a proof here. Most graduate-level linear algebra textbooks contain a proof of this result.)
Use Result A.18 (b) to prove

\[ A \text{ nonsingular } \iff |A| \neq 0. \]
Determinant of a Partitioned Matrix

If $A$ is nonsingular,

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A||D - CA^{-1}B|.$$ 

If $D$ is nonsingular,

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |D||A - BD^{-1}C|.$$
Proof:

\[
\begin{vmatrix}
A & B \\
C & D
\end{vmatrix} = \begin{vmatrix} I \end{vmatrix} \begin{vmatrix} A & B \\
C & D \end{vmatrix} = |AA^{-1}| \begin{vmatrix} A & B \\
C & D \end{vmatrix} = |A||A^{-1}| \begin{vmatrix} A & B \\
C & D \end{vmatrix}
\]

\[
= |A| \begin{vmatrix} A & B \\
C & D \end{vmatrix} |A^{-1}| = |A| \begin{vmatrix} A & B \\
C & D \end{vmatrix} A^{-1} - A^{-1}B
\]

\[
= |A| \begin{vmatrix} I & 0 \\
CA^{-1} & D - CA^{-1}B \end{vmatrix} = |A||D - CA^{-1}B|.
\]

A similar argument proves the second result. \qed