Determinants

Notation

The determinant of a square matrix $\underset{n \times n}{\boldsymbol{A}}$ is denoted $det(\boldsymbol{A})$ or $|\boldsymbol{A}|$.

Definition of Determinant

The determinant of $\underset{n\times n}{A} = [a_{ij}]$ is defined as

$$|\underset{n\times n}{A}| = \sum_{k=1}^{n!} (-1)^{m(p_k)} \prod_{i=1}^{n} a_{ip_k(i)},$$

where $p_1, \ldots, p_{n!}$ are the $n!$ distinct bijections (one-to-one and onto functions) from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$ and

$$m(p_k) = \sum_{i=1}^{n-1} \sum_{i^*=i+1}^{n} 1\!\!1[p_k(i) > p_k(i^*)].$$

Example: $n = 2$

Suppose

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Then $n! = 2$,

$$\begin{array}{lll} p_1(1) = 1 & p_1(2) = 2 & m(p_1) = 0 \\ p_2(1) = 2 & p_2(2) = 1 & m(p_2) = 1, \end{array}$$

and

$$\begin{aligned} |\boldsymbol{A}| &= \sum_{k=1}^{2} (-1)^{m(p_k)} \prod_{i=1}^{2} a_{ip_k(i)} \\ &= a_{11}a_{22} - a_{12}a_{21}. \end{aligned}$$

Example: $n = 3$

Find an expression for the determinant of

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Example: $n = 3$

$n! = 6$,

$$\begin{array}{llll} p_1(1) = 1 & p_1(2) = 2 & p_1(3) = 3 & m(p_1) = 0 \\ p_2(1) = 1 & p_2(2) = 3 & p_2(3) = 2 & m(p_2) = 1 \\ p_3(1) = 2 & p_3(2) = 1 & p_3(3) = 3 & m(p_3) = 1 \\ p_4(1) = 2 & p_4(2) = 3 & p_4(3) = 1 & m(p_4) = 2 \\ p_5(1) = 3 & p_5(2) = 1 & p_5(3) = 2 & m(p_5) = 2 \\ p_6(1) = 3 & p_6(2) = 2 & p_6(3) = 1 & m(p_6) = 3 \end{array}$$

, and

$$\begin{aligned} |\boldsymbol{A}| &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ &\quad + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}. \end{aligned}$$

Example: $n = 1$

Suppose $\boldsymbol{A} = [a_{11}]$. Then $n! = 1$,

$$p_1(1) = 1 \quad m(p_1) = 0,$$

and

$$|\boldsymbol{A}| = \sum_{k=1}^{1} (-1)^{m(p_k)} \prod_{i=1}^{1} a_{ip_k(i)} = a_{11}.$$

Verbal Description of Determinant Computation

1. Form all possible products of n matrix elements that contain exactly one element from each row and exactly one element from each column.
2. For each product, count the number of pairs of elements for which the "lower" element is to the "left" of the "higher" element. (For example, a_{31} is "lower" than a_{12} because a_{31} is in the third row while a_{12} is in the first row. Also, a_{31} is to the "left" of a_{12} because a_{31} is in the first column while a_{12} is in the second column.) If the number of such pairs is odd, multiply the product by -1. If the number of pairs is even, multiply the product by 1.
3. Add the signed products determined in steps 1 and 2.

Prove that if $\boldsymbol{A}$ is an $n \times n$ matrix and $c \in \mathbb{R}$, then

$$|c\boldsymbol{A}| = c^n|\boldsymbol{A}|.$$

Proof:

$$
\begin{aligned}
|c\mathbf{A}| &= \sum_{k=1}^{n!}(-1)^{m(p_k)}\prod_{i=1}^{n}(ca_{ip_k(i)}) \\
&= \sum_{k=1}^{n!}(-1)^{m(p_k)}c^n\prod_{i=1}^{n}a_{ip_k(i)} \\
&= c^n\sum_{k=1}^{n!}(-1)^{m(p_k)}\prod_{i=1}^{n}a_{ip_k(i)} \\
&= c^n|\mathbf{A}|.
\end{aligned}
$$

□

Prove that if $\underset{n\times n}{\boldsymbol{A}}$ is lower or upper triangular, then

$$|\boldsymbol{A}| = \prod_{i=1}^{n} a_{ii}.$$

(Note that this result implies $|\boldsymbol{A}| = \prod_{i=1}^{n} a_{ii}$ for a diagonal matrix.)

Proof for $\boldsymbol{A}$ Lower Triangular:

Suppose $\boldsymbol{A}$ is lower triangular. Then $a_{ij} = 0$ whenever $i < j$.

Thus, $\prod_{i=1}^{n} a_{ip_k(i)} = 0$ if $\exists\, i \ni p_k(i) > i$.

Because p_k is a bijection,

$$p_k(i) \leq i \;\forall\; i = 1, \ldots, n \Rightarrow p_k(i) = i \;\forall\; i = 1, \ldots, n.$$

Therefore,

$$|\boldsymbol{A}| = \sum_{k=1}^{n!} (-1)^{m(p_k)} \prod_{i=1}^{n} a_{ip_k(i)} = \prod_{i=1}^{n} a_{ii}.$$

□

Suppose $\boldsymbol{A}$ is an $n \times n$ matrix. Prove that $|\boldsymbol{A}| = |\boldsymbol{A}'|$.

Proof:

Let $q_k = p_k^{-1}$; i.e., $q_k(j) = i \iff p_k(i) = j$.

Note that

$$i < i^*,\ p_k(i) = j > p_k(i^*) = j^* \iff j^* < j,\ q_k(j^*) = i^* > q_k(j) = i.$$

Thus, $m(p_k) = m(q_k)$.

Proof:

It follows that

$$\begin{aligned}|\boldsymbol{A}| &= \sum_{k=1}^{n!}(-1)^{m(p_k)}\prod_{i=1}^{n} a_{ip_k(i)} \\ &= \sum_{k=1}^{n!}(-1)^{m(p_k)}\prod_{j=1}^{n} a_{q_k(j)j} \\ &= \sum_{k=1}^{n!}(-1)^{m(q_k)}\prod_{j=1}^{n} a_{q_k(j)j} \\ &= |\boldsymbol{A}'|.\end{aligned}$$

□

Find an expression for

$$\left| \begin{bmatrix} \underset{n\times n}{\boldsymbol{I}} & \underset{n\times m}{\boldsymbol{0}} \\ \underset{m\times n}{\boldsymbol{B}} & \underset{m\times m}{\boldsymbol{C}} \end{bmatrix} \right| = \begin{vmatrix} \underset{n\times n}{\boldsymbol{I}} & \underset{n\times m}{\boldsymbol{0}} \\ \underset{m\times n}{\boldsymbol{B}} & \underset{m\times m}{\boldsymbol{C}} \end{vmatrix}.$$

$$\begin{vmatrix} \underset{n\times n}{\boldsymbol{I}} & \underset{n\times m}{\boldsymbol{0}} \\ \underset{m\times n}{\boldsymbol{B}} & \underset{m\times m}{\boldsymbol{C}} \end{vmatrix} = |\boldsymbol{C}|$$

Likewise

$$\begin{vmatrix} \underset{n\times n}{\boldsymbol{B}} & \underset{n\times m}{\boldsymbol{0}} \\ \underset{m\times n}{\boldsymbol{C}} & \underset{m\times m}{\boldsymbol{I}} \end{vmatrix} = |\boldsymbol{B}|,$$

$$\begin{vmatrix} \underset{n\times n}{\boldsymbol{B}} & \underset{n\times m}{\boldsymbol{C}} \\ \underset{m\times n}{\boldsymbol{0}} & \underset{m\times m}{\boldsymbol{I}} \end{vmatrix} = |\boldsymbol{B}|,$$

and

$$\begin{vmatrix} \underset{n\times n}{\boldsymbol{I}} & \underset{n\times m}{\boldsymbol{B}} \\ \underset{m\times n}{\boldsymbol{0}} & \underset{m\times m}{\boldsymbol{C}} \end{vmatrix} = |\boldsymbol{C}|.$$

More generally, it can be shown that

$$\left| \begin{array}{cc} \underset{n\times n}{\boldsymbol{B}} & \underset{n\times m}{\boldsymbol{0}} \\ \underset{m\times n}{\boldsymbol{C}} & \underset{m\times m}{\boldsymbol{D}} \end{array} \right| = |\boldsymbol{B}||\boldsymbol{D}|$$

and

$$\left| \begin{array}{cc} \underset{n\times n}{\boldsymbol{B}} & \underset{n\times m}{\boldsymbol{C}} \\ \underset{m\times n}{\boldsymbol{0}} & \underset{m\times m}{\boldsymbol{D}} \end{array} \right| = |\boldsymbol{B}||\boldsymbol{D}|$$

Result A.18 (b):

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are each $n \times n$ matrices, then

$$|\boldsymbol{AB}| = |\boldsymbol{A}||\boldsymbol{B}|.$$

(We will not go through a proof here. Most graduate-level linear algebra textbooks contain a proof of this result.)

Use Result A.18 (b) to prove

$$\underset{n\times n}{\boldsymbol{A}} \text{ nonsingular} \iff |\boldsymbol{A}| \neq 0.$$

Proof ($\Longrightarrow$):

$$|\boldsymbol{A}||\boldsymbol{A}^{-1}| = |\boldsymbol{A}\boldsymbol{A}^{-1}| = |\boldsymbol{I}| = 1 \Rightarrow |\boldsymbol{A}| \neq 0.$$

Proof ($\Longleftarrow$):

Suppose $\boldsymbol{A}$ is singular. We will show $|\boldsymbol{A}| = 0$.

$\boldsymbol{A}$ singular $\Rightarrow \;\exists\, \boldsymbol{b} \neq \boldsymbol{0} \;\ni\; \boldsymbol{A}\boldsymbol{b} = \boldsymbol{0}$.

Let $\underset{n\times n}{\boldsymbol{B}}$ be a nonsingular matrix whose first column is $\boldsymbol{b}$. (We know such a matrix exists by Fact V5.)

Then $|\boldsymbol{A}||\boldsymbol{B}| = |\boldsymbol{A}\boldsymbol{B}| = 0$ because the first column of $\boldsymbol{A}\boldsymbol{B}$ is $\boldsymbol{0}$.

Dividing both sides by $|\boldsymbol{B}| \neq 0$ yields $|\boldsymbol{A}| = 0$. $\square$

Determinant of a Partitioned Matrix

If $\boldsymbol{A}$ is nonsingular,

$$\begin{vmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{vmatrix} = |\boldsymbol{A}||\boldsymbol{D} - \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B}|.$$

If $\boldsymbol{D}$ is nonsingular,

$$\begin{vmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{vmatrix} = |\boldsymbol{D}||\boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C}|.$$

Proof:

$$\begin{aligned}
\begin{vmatrix} A & B \\ C & D \end{vmatrix} &= |I| \begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AA^{-1}| \begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A||A^{-1}| \begin{vmatrix} A & B \\ C & D \end{vmatrix} \\
&= |A| \begin{vmatrix} A & B \\ C & D \end{vmatrix} |A^{-1}| = |A| \left| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} A^{-1} & -A^{-1}B \\ 0 & I \end{bmatrix} \right| \\
&= |A| \begin{vmatrix} I & 0 \\ CA^{-1} & D - CA^{-1}B \end{vmatrix} = |A||D - CA^{-1}B|.
\end{aligned}$$

A similar argument proves the second result. □