Best Linear Unbiased Estimation in Constrained Models

Suppose the GMM holds and suppose $\boldsymbol{\beta} \in \mathbb{R}^p$ satisfies $\boldsymbol{H}'\boldsymbol{\beta} = \boldsymbol{h}$ for some known $\underset{p \times q}{\boldsymbol{H}}$ of rank q and some known $\underset{q \times 1}{\boldsymbol{h}}$.

The purpose of these notes is to show that if $\tilde{\beta}$ is the first component of a solution to the RNE

$$\begin{bmatrix} X'X & H \\ H' & 0 \end{bmatrix} \begin{bmatrix} b \\ \lambda \end{bmatrix} = \begin{bmatrix} X'y \\ h \end{bmatrix},$$

then $c'\tilde{\beta}$ is the BLUE of estimable $c'\beta$ under the constrained GMM (CGMM).

Lemma 4.2:

If $\boldsymbol{c}'\boldsymbol{\beta}$ is estimable under the constrained model, then the following equations have a solution

$$\begin{bmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{H} \\ \boldsymbol{H}' & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{0} \end{bmatrix}.$$

Proof of Lemma 4.2:

By Result 3.7, $\boldsymbol{c}'\boldsymbol{\beta}$ estimable under the constrained model
$\Rightarrow \exists\ \boldsymbol{a}$ and $\boldsymbol{l} \ni \boldsymbol{c} = \boldsymbol{X}'\boldsymbol{a} + \boldsymbol{H}\boldsymbol{l}$. (Fact 1)

By Result 3.8, the RNE are consistent.

Thus, $\exists$ a solution to

$$\begin{bmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{H} \\ \boldsymbol{H}' & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}'\boldsymbol{y} \\ \boldsymbol{h} \end{bmatrix}$$

$\forall\ \boldsymbol{y}$ and $\forall\ \boldsymbol{h} \ni \boldsymbol{H}'\boldsymbol{b} = \boldsymbol{h}$ is consistent. (Fact 2)

By Fact 2, $\exists\ \boldsymbol{b}^*$ and $\boldsymbol{\lambda}^* \ni$

$$\begin{bmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{H} \\ \boldsymbol{H}' & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{b}^* \\ \boldsymbol{\lambda}^* \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}'\boldsymbol{a} \\ \boldsymbol{0} \end{bmatrix}.$$

Thus, $\boldsymbol{X}'\boldsymbol{X}\boldsymbol{b}^* + \boldsymbol{H}\boldsymbol{\lambda}^* = \boldsymbol{X}'\boldsymbol{a}$ and $\boldsymbol{H}'\boldsymbol{b}^* = \boldsymbol{0}$.

Using this with Fact 1 $\Rightarrow$

$$\begin{aligned} \boldsymbol{c} &= \boldsymbol{X}'\boldsymbol{X}\boldsymbol{b}^* + \boldsymbol{H}\boldsymbol{\lambda}^* + \boldsymbol{H}\boldsymbol{l} \quad \text{for some } \boldsymbol{l} \\ &= \boldsymbol{X}'\boldsymbol{X}\boldsymbol{b}^* + \boldsymbol{H}(\boldsymbol{\lambda}^* + \boldsymbol{l}) \quad \text{and} \\ \boldsymbol{H}'\boldsymbol{b}^* &= \boldsymbol{0}. \end{aligned}$$

$$\therefore \begin{bmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{H} \\ \boldsymbol{H}' & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{b}^* \\ \boldsymbol{\lambda}^* + \boldsymbol{l} \end{bmatrix} = \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{0} \end{bmatrix}.$$

$\therefore$ There exists a solution to $\begin{bmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{H} \\ \boldsymbol{H}' & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{0} \end{bmatrix}.$

$\square$

Lemma 4.3:

If $\tilde{\boldsymbol{\beta}}$ is the first part of a solution to the RNE and if $\boldsymbol{c}'\boldsymbol{\beta}$ is estimable, then $\boldsymbol{c}'\tilde{\boldsymbol{\beta}}$ is an unbiased linear estimator of $\boldsymbol{c}'\boldsymbol{\beta}$ under the restricted model.

Proof of Lemma 4.3:

By Lemma 4.2, $\exists\ \boldsymbol{v}_1, \boldsymbol{v}_2 \ni$

$$\begin{bmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{H} \\ \boldsymbol{H}' & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{0} \end{bmatrix} \iff$$

$$\boldsymbol{X}'\boldsymbol{X}\boldsymbol{v}_1 + \boldsymbol{H}\boldsymbol{v}_2 = \boldsymbol{c} \quad \text{and} \tag{1}$$

$$\boldsymbol{H}'\boldsymbol{v}_1 = \boldsymbol{0}. \tag{2}$$

Thus,

$$\begin{aligned} \boldsymbol{c}'\tilde{\boldsymbol{\beta}} &= (\boldsymbol{v}_1'\boldsymbol{X}'\boldsymbol{X} + \boldsymbol{v}_2'\boldsymbol{H}')\tilde{\boldsymbol{\beta}} \\ &= \boldsymbol{v}_1'\boldsymbol{X}'\boldsymbol{X}\tilde{\boldsymbol{\beta}} + \boldsymbol{v}_2'\boldsymbol{H}'\tilde{\boldsymbol{\beta}}. \end{aligned} \tag{3}$$

Now $\tilde{\beta}$ a leading subvector of a solution to the RNE

$$\iff \begin{bmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{H} \\ \boldsymbol{H}' & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{\beta}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}'\boldsymbol{y} \\ \boldsymbol{h} \end{bmatrix} \quad \text{for some } \boldsymbol{\lambda}$$

$$\iff \boldsymbol{X}'\boldsymbol{X}\tilde{\boldsymbol{\beta}} + \boldsymbol{H}\boldsymbol{\lambda} = \boldsymbol{X}'\boldsymbol{y} \quad \text{for some } \boldsymbol{\lambda} \tag{4}$$

$$\text{and} \quad \boldsymbol{H}'\tilde{\boldsymbol{\beta}} = \boldsymbol{h}. \tag{5}$$

It follows that

$$
\begin{aligned}
\boldsymbol{c}'\tilde{\boldsymbol{\beta}} &= \boldsymbol{v}_1'(\boldsymbol{X}'\boldsymbol{y} - \boldsymbol{H}\boldsymbol{\lambda}) + \boldsymbol{v}_2'\boldsymbol{h} \quad \text{(by (3), (4), (5))} \\
&= \boldsymbol{v}_1'\boldsymbol{X}'\boldsymbol{y} - \boldsymbol{v}_1'\boldsymbol{H}\boldsymbol{\lambda} + \boldsymbol{v}_2'\boldsymbol{h} \\
&= \boldsymbol{v}_1'\boldsymbol{X}'\boldsymbol{y} + \boldsymbol{v}_2'\boldsymbol{h}. \quad \text{(by (2))}
\end{aligned}
$$

$$
\begin{aligned}
E(\boldsymbol{c}'\tilde{\boldsymbol{\beta}}) &= \boldsymbol{v}_1'\boldsymbol{X}'E(\boldsymbol{y}) + \boldsymbol{v}_2'\boldsymbol{h} \\
&= \boldsymbol{v}_1'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{v}_2'\boldsymbol{h} \\
&= (\boldsymbol{c}' - \boldsymbol{v}_2'\boldsymbol{H}')\boldsymbol{\beta} + \boldsymbol{v}_2'\boldsymbol{h} \quad \text{(by (1))} \\
&= \boldsymbol{c}'\boldsymbol{\beta} - \boldsymbol{v}_2'\boldsymbol{H}'\boldsymbol{\beta} + \boldsymbol{v}_2'\boldsymbol{h} \\
&= \boldsymbol{c}'\boldsymbol{\beta} \quad \forall\ \boldsymbol{\beta} \ni \boldsymbol{H}'\boldsymbol{\beta} = \boldsymbol{h}.
\end{aligned}
$$

$\square$

Result 4.5:

Suppose the CGMM holds, $\tilde{\boldsymbol{\beta}}$ is the leading subvector of a solution to the RNE, and $\boldsymbol{c}'\boldsymbol{\beta}$ is estimable under the constrained model. Then $\boldsymbol{c}'\tilde{\boldsymbol{\beta}}$ is the BLUE of $\boldsymbol{c}'\boldsymbol{\beta}$ under the constrained model.

Proof of Result 4.5:

By Result 3.7, any linear estimator unbiased for $\boldsymbol{c}'\boldsymbol{\beta}$ under the constrained model has the form

$$\boldsymbol{l}'\boldsymbol{h} + \boldsymbol{a}'\boldsymbol{y},$$

where

$$\boldsymbol{c} = \boldsymbol{X}'\boldsymbol{a} + \boldsymbol{H}\boldsymbol{l}. \tag{6}$$

$$
\begin{aligned}
\text{Var}(\boldsymbol{l}'\boldsymbol{h} + \boldsymbol{a}'\boldsymbol{y}) &= \text{Var}(\boldsymbol{l}'\boldsymbol{h} + \boldsymbol{a}'\boldsymbol{y} - \boldsymbol{c}'\tilde{\boldsymbol{\beta}} + \boldsymbol{c}'\tilde{\boldsymbol{\beta}}) \\
&= \text{Var}(\boldsymbol{l}'\boldsymbol{h} + \boldsymbol{a}'\boldsymbol{y} - \boldsymbol{c}'\tilde{\boldsymbol{\beta}}) + \text{Var}(\boldsymbol{c}'\tilde{\boldsymbol{\beta}}) \\
&\quad + 2\text{Cov}(\boldsymbol{l}'\boldsymbol{h} + \boldsymbol{a}'\boldsymbol{y} - \boldsymbol{c}'\tilde{\boldsymbol{\beta}}, \boldsymbol{c}'\tilde{\boldsymbol{\beta}}).
\end{aligned}
$$

Substituting $\boldsymbol{v}_1'\boldsymbol{X}'\boldsymbol{y} + \boldsymbol{v}_2'\boldsymbol{h}$ for $\boldsymbol{c}'\tilde{\boldsymbol{\beta}}$ (see slide 11) leads to the following expression for the covariance:

$$
\begin{aligned}
\text{Cov}(\boldsymbol{a}'\boldsymbol{y} - \boldsymbol{v}_1'\boldsymbol{X}'\boldsymbol{y}, \boldsymbol{v}_1'\boldsymbol{X}'\boldsymbol{y}) &= \text{Cov}((\boldsymbol{a}' - \boldsymbol{v}_1'\boldsymbol{X}')\boldsymbol{y}, \boldsymbol{v}_1'\boldsymbol{X}'\boldsymbol{y}) \\
&= \sigma^2(\boldsymbol{a}' - \boldsymbol{v}_1'\boldsymbol{X}')\boldsymbol{X}\boldsymbol{v}_1 \\
&= \sigma^2(\boldsymbol{a}'\boldsymbol{X} - \boldsymbol{v}_1'\boldsymbol{X}'\boldsymbol{X})\boldsymbol{v}_1 \\
&= \sigma^2((\boldsymbol{c}' - \boldsymbol{l}'\boldsymbol{H}') - (\boldsymbol{c}' - \boldsymbol{v}_2'\boldsymbol{H}'))\boldsymbol{v}_1 \quad \text{(by (6), (1))} \\
&= \sigma^2(\boldsymbol{v}_2' - \boldsymbol{l}')\boldsymbol{H}'\boldsymbol{v}_1 \\
&= \sigma^2(\boldsymbol{v}_2 - \boldsymbol{l})'\boldsymbol{0} = 0. \quad \text{(by (2))}
\end{aligned}
$$

Thus, $\mathrm{Var}(\boldsymbol{l}'\boldsymbol{h} + \boldsymbol{a}'\boldsymbol{y}) \geq \mathrm{Var}(\boldsymbol{c}'\tilde{\boldsymbol{\beta}})$ with equality iff

$$\begin{aligned}
&\mathrm{Var}(\boldsymbol{l}'\boldsymbol{h} + \boldsymbol{a}'\boldsymbol{y} - \boldsymbol{c}'\tilde{\boldsymbol{\beta}}) = 0 \\
&\iff \mathrm{Var}(\boldsymbol{a}'\boldsymbol{y} - \boldsymbol{c}'\tilde{\boldsymbol{\beta}}) = 0 \\
&\iff \mathrm{Var}(\boldsymbol{a}'\boldsymbol{y} - \boldsymbol{v}_1'\boldsymbol{X}'\boldsymbol{y} - \boldsymbol{v}_2'\boldsymbol{h}) = 0 \\
&\iff \mathrm{Var}(\boldsymbol{a}'\boldsymbol{y} - \boldsymbol{v}_1'\boldsymbol{X}'\boldsymbol{y}) = 0 \\
&\iff \mathrm{Var}((\boldsymbol{a} - \boldsymbol{X}\boldsymbol{v}_1)'\boldsymbol{y}) = 0 \\
&\iff \sigma^2(\boldsymbol{a} - \boldsymbol{X}\boldsymbol{v}_1)'(\boldsymbol{a} - \boldsymbol{X}\boldsymbol{v}_1) = 0 \\
&\iff \boldsymbol{a} = \boldsymbol{X}\boldsymbol{v}_1 \\
&\iff \boldsymbol{a} = \boldsymbol{X}\boldsymbol{v}_1 \quad \text{and} \quad \boldsymbol{H}\boldsymbol{v}_2 = \boldsymbol{H}\boldsymbol{l} \text{ because...}
\end{aligned}$$

$$\boldsymbol{c} = \boldsymbol{X}'\boldsymbol{a} + \boldsymbol{H}\boldsymbol{l} = \boldsymbol{X}'\boldsymbol{X}\boldsymbol{v}_1 + \boldsymbol{H}\boldsymbol{v}_2,$$

by (1) and (6) so that $\boldsymbol{a} = \boldsymbol{X}\boldsymbol{v}_1 \Rightarrow \boldsymbol{H}\boldsymbol{l} = \boldsymbol{H}\boldsymbol{v}_2$.

Recall that $\boldsymbol{H}$ is of full-column rank. Thus

$$\begin{aligned} \boldsymbol{H}\boldsymbol{l} = \boldsymbol{H}\boldsymbol{v}_2 &\iff \boldsymbol{H}(\boldsymbol{l} - \boldsymbol{v}_2) = \boldsymbol{0} \\ &\iff \boldsymbol{l} - \boldsymbol{v}_2 = \boldsymbol{0} \\ &\iff \boldsymbol{l} = \boldsymbol{v}_2. \end{aligned}$$

It follows that

$$\text{Var}(\boldsymbol{l}'\boldsymbol{h} + \boldsymbol{a}'\boldsymbol{y}) \geq \text{Var}(\boldsymbol{c}'\tilde{\boldsymbol{\beta}})$$

with equality iff

$$\begin{aligned} \boldsymbol{l}'\boldsymbol{h} + \boldsymbol{a}'\boldsymbol{y} &= \boldsymbol{v}_1'\boldsymbol{X}'\boldsymbol{y} + \boldsymbol{v}_2'\boldsymbol{h} \\ &= \boldsymbol{c}'\tilde{\boldsymbol{\beta}}. \end{aligned}$$

$\therefore$ the constrained BLUE is unique. $\square$

Note that we can handle BLUE in the constrained version of the AM by transforming to the GMM.

The constraint is unaffected by transformation.

$$\begin{aligned} \boldsymbol{V}^{-1/2}\boldsymbol{y} &= \boldsymbol{V}^{-1/2}\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{V}^{-1/2}\boldsymbol{\varepsilon} \\ &= \boldsymbol{U}\boldsymbol{\beta} + \boldsymbol{\delta}. \end{aligned}$$