When is the OLSE the BLUE?

When is the Ordinary Least Squares Estimator (OLSE) the Best Linear Unbiased Estimator (BLUE)?

We already know the OLSE is the BLUE under the GMM, but are there other situations where the OLSE is the BLUE?

Consider an experiment involving 4 plants.

Two leaves are randomly selected from each plant.

One leaf from each plant is randomly selected for treatment with treatment 1.

The other leaf from each plant receives treatment 2.

Let y_{ij} = the response for the treatment i leaf from plant j $(i = 1, 2;\ j = 1, 2, 3, 4)$. Suppose

$$y_{ij} = \mu_i + p_j + e_{ij},$$

where $p_1, \ldots, p_4, e_{11}, \ldots, e_{24}$ are uncorrelated,

$$E(p_j) = E(e_{ij}) = 0, \mathrm{Var}(p_j) = \sigma_p^2, \mathrm{Var}(e_{ij}) = \sigma^2 \quad \forall\ i = 1, 2;\ j = 1, \ldots, 4.$$

Suppose σ_p^2/σ^2 is known to be equal to 2 and

$$\boldsymbol{y} = (y_{11}, y_{21}, y_{12}, y_{22}, y_{13}, y_{23}, y_{14}, y_{24})'.$$

Show that the AM holds.

Find OLSE pf $\mu_1 - \mu_2$ and the BLUE of $\mu_1 - \mu_2$.

$E(\boldsymbol{y}) = \boldsymbol{X}\boldsymbol{\beta}$, where

$$\boldsymbol{X} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\beta} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}.$$

$$
\begin{aligned}
\text{Var}(y_{ij}) &= \text{Var}(\mu_i + p_j + e_{ij}) \\
&= \text{Var}(p_j + e_{ij}) \\
&= \text{Var}(p_j) + \text{Var}(e_{ij}) \\
&= \sigma_p^2 + \sigma^2 \\
&= \sigma^2(\sigma_p^2/\sigma^2 + 1) \\
&= 3\sigma^2.
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Cov}(y_{1j}, y_{2j}) &= \operatorname{Cov}(\mu_1 + p_j + e_{1j}, \mu_2 + p_j + e_{2j}) \\
&= \operatorname{Cov}(p_j + e_{1j}, p_j + e_{2j}) \\
&= \operatorname{Cov}(p_j, p_j) + \operatorname{Cov}(p_j, e_{2j}) + \operatorname{Cov}(p_j, e_{1j}) + \operatorname{Cov}(e_{1j}, e_{2j}) \\
&= \operatorname{Cov}(p_j, p_j) \\
&= \operatorname{Var}(p_j) = \sigma_p^2 \\
&= \sigma^2(\sigma_p^2/\sigma^2) = 2\sigma^2.
\end{aligned}
$$

$$
\begin{aligned}
\text{Cov}(y_{ij}, y_{i^*j^*}) &= 0 \quad \text{if} \quad j \neq j^* \quad \text{because} \\
&= \text{Cov}(p_j + e_{ij}, p_{j^*} + e_{i^*j^*}) \\
&= \text{Cov}(p_j, p_{j^*}) + \text{Cov}(p_j, e_{i^*j^*}) \\
&\quad + \text{Cov}(p_{j^*}, e_{ij}) + \text{Cov}(e_{ij}, e_{i^*j^*}) \\
&= 0.
\end{aligned}
$$

Thus, $\text{Var}(\boldsymbol{y}) = \sigma^2 \boldsymbol{V}$, where

$$\boldsymbol{V} = \begin{bmatrix} 3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 \end{bmatrix}.$$

We can write the model as

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon},$$

where

$$\boldsymbol{\varepsilon} = \begin{bmatrix} p_1 + e_{11} \\ p_1 + e_{21} \\ p_2 + e_{12} \\ p_2 + e_{22} \\ p_3 + e_{13} \\ p_3 + e_{23} \\ p_4 + e_{14} \\ p_4 + e_{24} \end{bmatrix}, \quad \boldsymbol{E}(\boldsymbol{\varepsilon}) = \boldsymbol{0} \quad \text{and} \quad \text{Var}(\boldsymbol{\varepsilon}) = \sigma^2 \boldsymbol{V}.$$

Note that

$$\boldsymbol{X} = \begin{bmatrix} \underset{2\times 2}{\boldsymbol{I}} \\ \underset{2\times 2}{\boldsymbol{I}} \\ \underset{2\times 2}{\boldsymbol{I}} \\ \underset{2\times 2}{\boldsymbol{I}} \end{bmatrix} = \underset{4\times 1}{\mathbf{1}} \otimes \underset{2\times 2}{\boldsymbol{I}}.$$

Thus,

$$X'X = [I, I, I, I] \begin{bmatrix} I \\ I \\ I \\ I \end{bmatrix} = 4 \underset{2\times 2}{I}$$

$$(X'X)^{-1} = 1/4I.$$

$$\boldsymbol{X}'\boldsymbol{y} = [\boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}]\boldsymbol{y} = \begin{bmatrix} y_{1\cdot} \\ y_{2\cdot} \end{bmatrix}.$$

Thus,

$$\hat{\boldsymbol{\beta}}_{\text{OLS}} = \frac{1}{4}\boldsymbol{I} \begin{bmatrix} y_{1\cdot} \\ y_{2\cdot} \end{bmatrix} = \begin{bmatrix} \bar{y}_{1\cdot} \\ \bar{y}_{2\cdot} \end{bmatrix}.$$

Thus, the OLSE of $\mu_1 - \mu_2$ is $[1, -1]\hat{\boldsymbol{\beta}}_{\text{OLS}} = \bar{y}_{1\cdot} - \bar{y}_{2\cdot}$.

To find the GLSE, which we know is the BLUE for this AM, let $\boldsymbol{A} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$. Then

$$\boldsymbol{V} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A} \end{bmatrix} = \underset{4\times 4}{\boldsymbol{I}} \otimes \boldsymbol{A}.$$

$$\boldsymbol{V}^{-1} = \underset{4\times 4}{\boldsymbol{I}} \otimes \boldsymbol{A}^{-1} = \begin{bmatrix} \boldsymbol{A}^{-1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}^{-1} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A}^{-1} \end{bmatrix}$$

It follows that

$$\begin{aligned}
\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X} &= [\mathbf{1}' \otimes \boldsymbol{I}][\boldsymbol{I} \otimes \boldsymbol{A}^{-1}][\mathbf{1} \times \boldsymbol{I}] \\
&= [\boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}] \begin{bmatrix} \boldsymbol{A}^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{A}^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{A}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{A}^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{I} \\ \boldsymbol{I} \\ \boldsymbol{I} \end{bmatrix} \\
&= 4\boldsymbol{A}^{-1}.
\end{aligned}$$

Thus, $(\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-1} = \frac{1}{4}\boldsymbol{A}$.

$$X'V^{-1}y = [I, I, I, I] \begin{bmatrix} A^{-1} & 0 & 0 & 0 \\ 0 & A^{-1} & 0 & 0 \\ 0 & 0 & A^{-1} & 0 \\ 0 & 0 & 0 & A^{-1} \end{bmatrix} y$$

$$= [A^{-1},\ A^{-1},\ A^{-1},\ A^{-1}]y$$

$$= A^{-1}[I, I, I, I]y.$$

Thus,

$$\begin{aligned}\hat{\beta}_{\text{GLS}} &= \frac{1}{4}\boldsymbol{A}\boldsymbol{A}^{-1}[\boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}]\boldsymbol{y} \\ &= \frac{1}{4}[\boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}]\boldsymbol{y} \\ &= \begin{bmatrix} \bar{y}_{1\cdot} \\ \bar{y}_{2\cdot} \end{bmatrix} = \hat{\beta}_{\text{OLS}}.\end{aligned}$$

Thus, the GLSE of $\mu_1 - \mu_2$ and the BLUE of $\mu_1 - \mu_2$ is $[1, -1]\hat{\beta}_{\text{GLS}} = \bar{y}_{1\cdot} - \bar{y}_{2\cdot}$.

Thus,

$$\text{OLSE} = \text{GLSE} = \text{BLUE}$$

in this case.

Although we assumed that $\sigma_p^2/\sigma^2 = 2$ in our example, that assumption was not needed to find the GLSE.

We have looked at one specific example where the OLSE = GLSE = BLUE.

What general conditions must be satisfied for this to hold?

Result 4.3:

Suppose the AM holds. The estimator $\boldsymbol{t}'\boldsymbol{y}$ is the BLUE for $E(\boldsymbol{t}'\boldsymbol{y}) \iff \boldsymbol{t}'\boldsymbol{y}$ is uncorrelated with all linear unbiased estimators of zero.

Proof:

($\Longleftarrow$) Let $\boldsymbol{h}'\boldsymbol{y}$ be an arbitrary linear unbiased estimator of $E(\boldsymbol{t}'\boldsymbol{y})$. Then

$$\begin{aligned} E((\boldsymbol{h}-\boldsymbol{t})'\boldsymbol{y}) &= E(\boldsymbol{h}'\boldsymbol{y}-\boldsymbol{t}'\boldsymbol{y}) \\ &= E(\boldsymbol{h}'\boldsymbol{y}) - E(\boldsymbol{t}'\boldsymbol{y}) \\ &= 0. \end{aligned}$$

Thus, $(\boldsymbol{h}-\boldsymbol{t})'\boldsymbol{y}$ is linear unbiased for zero.

It follows that

$$\text{Cov}(\boldsymbol{t}'\boldsymbol{y}, (\boldsymbol{h}-\boldsymbol{t})'\boldsymbol{y}) = 0.$$

$$\begin{aligned}\text{Var}(\boldsymbol{h}'\boldsymbol{y}) &= \text{Var}(\boldsymbol{h}'\boldsymbol{y} - \boldsymbol{t}'\boldsymbol{y} + \boldsymbol{t}'\boldsymbol{y}) \\ &= \text{Var}((\boldsymbol{h}-\boldsymbol{t})'\boldsymbol{y}) + \text{Var}(\boldsymbol{t}'\boldsymbol{y}).\end{aligned}$$

$\therefore \text{Var}(\boldsymbol{h}'\boldsymbol{y}) \geq \text{Var}(\boldsymbol{t}'\boldsymbol{y})$ with equality iff

$$\text{Var}((\boldsymbol{h}-\boldsymbol{t})'\boldsymbol{y}) = 0 \iff \boldsymbol{h} = \boldsymbol{t}.$$

$\therefore \boldsymbol{t}'\boldsymbol{y}$ is the BLUE of $E(\boldsymbol{t}'\boldsymbol{y})$.

$(\Longrightarrow)$ Suppose $\boldsymbol{h}'\boldsymbol{y}$ is a linear unbiased estimator of zero. If $\boldsymbol{h} = \boldsymbol{0}$, then

$$\mathrm{Cov}(\boldsymbol{t}'\boldsymbol{y}, \boldsymbol{h}'\boldsymbol{y}) = \mathrm{Cov}(\boldsymbol{t}'\boldsymbol{y}, 0) = 0.$$

Now suppose $\boldsymbol{h} \neq \boldsymbol{0}$. Let

$$c = \mathrm{Cov}(\boldsymbol{t}'\boldsymbol{y}, \boldsymbol{h}'\boldsymbol{y}) \quad \text{and} \quad d = \mathrm{Var}(\boldsymbol{h}'\boldsymbol{y}) > 0.$$

We need to show $c = 0$.

Now consider $\boldsymbol{a}'\boldsymbol{y} = \boldsymbol{t}'\boldsymbol{y} - (c/d)\boldsymbol{h}'\boldsymbol{y}$.

$$\begin{aligned} E(\boldsymbol{a}'\boldsymbol{y}) &= E(\boldsymbol{t}'\boldsymbol{y}) - (c/d)E(\boldsymbol{h}'\boldsymbol{y}) \\ &= E(\boldsymbol{t}'\boldsymbol{y}). \end{aligned}$$

Thus, $\boldsymbol{a}'\boldsymbol{y}$ is a linear unbiased estimator of $E(\boldsymbol{t}'\boldsymbol{y})$.

$$
\begin{aligned}
\text{Var}(\boldsymbol{a}'\boldsymbol{y}) &= \text{Var}(\boldsymbol{t}'\boldsymbol{y} - (c/d)\boldsymbol{h}'\boldsymbol{y}) \\
&= \text{Var}(\boldsymbol{t}'\boldsymbol{y}) + \frac{c^2}{d^2}\text{Var}(\boldsymbol{h}'\boldsymbol{y}) \\
&\quad - 2\text{Cov}(\boldsymbol{t}'\boldsymbol{y}, (c/d)\boldsymbol{h}'\boldsymbol{y}) \\
&= \text{Var}(\boldsymbol{t}'\boldsymbol{y}) + \frac{c^2}{d^2}d - 2(c/d)c \\
&= \text{Var}(\boldsymbol{t}'\boldsymbol{y}) - \frac{c^2}{d}.
\end{aligned}
$$

Now

$$\mathrm{Var}(\boldsymbol{a}'\boldsymbol{y}) = \mathrm{Var}(\boldsymbol{t}'\boldsymbol{y}) - \frac{c^2}{d}$$

$\Rightarrow c = 0 \because \boldsymbol{t}'\boldsymbol{y}$ has lowest variance among all linear unbiased estimator of $E(\boldsymbol{t}'\boldsymbol{y})$.

□

Corollary 4.1:

Under the AM, the estimator $\boldsymbol{t}'\boldsymbol{y}$ is the BLUE of $E(\boldsymbol{t}'\boldsymbol{y}) \iff \boldsymbol{V}\boldsymbol{t} \in \mathcal{C}(\boldsymbol{X})$.

Proof of Corollary 4.1:

First note that $\boldsymbol{h}'\boldsymbol{y}$ a linear unbiased estimator of zero is equivalent to

$$E(\boldsymbol{h}'\boldsymbol{y}) = 0 \quad \forall\, \boldsymbol{\beta} \in \mathbb{R}^p$$
$$\iff \boldsymbol{h}'\boldsymbol{X}\boldsymbol{\beta} = 0 \quad \forall\, \boldsymbol{\beta} \in \mathbb{R}^p$$
$$\iff \boldsymbol{h}'\boldsymbol{X} = \boldsymbol{0}' \iff \boldsymbol{X}'\boldsymbol{h} = \boldsymbol{0} \iff \boldsymbol{h} \in \mathcal{N}(\boldsymbol{X}').$$

Thus, by Result 4.3,

$$
\begin{aligned}
&\boldsymbol{t}'\boldsymbol{y} \text{ BLUE for } E(\boldsymbol{t}'\boldsymbol{y}) \\
&\iff \mathrm{Cov}(\boldsymbol{h}'\boldsymbol{y}, \boldsymbol{t}'\boldsymbol{y}) = 0 \quad \forall\, \boldsymbol{h} \in \mathcal{N}(\boldsymbol{X}') \\
&\iff \sigma^2 \boldsymbol{h}'\boldsymbol{V}\boldsymbol{t} = 0 \quad \forall\, \boldsymbol{h} \in \mathcal{N}(\boldsymbol{X}') \\
&\iff \boldsymbol{h}'\boldsymbol{V}\boldsymbol{t} = 0 \quad \forall\, \boldsymbol{h} \in \mathcal{N}(\boldsymbol{X}') \\
&\iff \boldsymbol{V}\boldsymbol{t} \in \mathcal{N}(\boldsymbol{X}')^{\perp} = \mathcal{C}(\boldsymbol{X}).
\end{aligned}
$$

□

Result 4.4:

Under the AM, the OLSE of $\boldsymbol{c}'\boldsymbol{\beta}$ is the BLUE of $\boldsymbol{c}'\boldsymbol{\beta}$ $\forall$ estimable $\boldsymbol{c}'\boldsymbol{\beta} \iff \exists$ a matrix $\boldsymbol{Q} \ni \boldsymbol{V}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{Q}$.

Proof of Result 4.4:

($\Longleftarrow$) Suppose $\boldsymbol{c}'\boldsymbol{\beta}$ is estimable.

Let $\boldsymbol{t}' = \boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'$. Then

$$\begin{aligned}
&\boldsymbol{V}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{Q} \\
&\Rightarrow \boldsymbol{V}\boldsymbol{X}[(\boldsymbol{X}'\boldsymbol{X})^{-}]'\boldsymbol{c} = \boldsymbol{X}\boldsymbol{Q}[(\boldsymbol{X}'\boldsymbol{X})^{-}]'\boldsymbol{c} \\
&\Rightarrow \boldsymbol{V}\boldsymbol{t} \in \mathcal{C}(\boldsymbol{X}), \text{ which by Cor. 4.1,} \\
&\Rightarrow \boldsymbol{t}'\boldsymbol{y} \text{ is BLUE of } E(\boldsymbol{t}'\boldsymbol{y}) \\
&\Rightarrow \boldsymbol{c}'\hat{\boldsymbol{\beta}}_{\text{OLS}} \text{ is BLUE of } \boldsymbol{c}'\boldsymbol{\beta}.
\end{aligned}$$

$(\Longrightarrow)$ By Corollary 4.1, $\boldsymbol{c}'\hat{\boldsymbol{\beta}}_{\text{OLS}}$ is BLUE for any estimable $\boldsymbol{c}'\boldsymbol{\beta}$.

$$\Rightarrow \boldsymbol{V}\boldsymbol{X}[(\boldsymbol{X}'\boldsymbol{X})^{-}]'\boldsymbol{c} \in \mathcal{C}(\boldsymbol{X}) \quad \forall\, \boldsymbol{c} \in \mathcal{C}(\boldsymbol{X}')$$

$$\Rightarrow \boldsymbol{V}\boldsymbol{X}[(\boldsymbol{X}'\boldsymbol{X})^{-}]'\boldsymbol{X}'\boldsymbol{a} \in \mathcal{C}(\boldsymbol{X}) \quad \forall\, \boldsymbol{a} \in \mathbb{R}^n$$

$$\Rightarrow \boldsymbol{V}\boldsymbol{P}_{\boldsymbol{X}}\boldsymbol{a} \in \mathcal{C}(\boldsymbol{X}) \quad \forall\, \boldsymbol{a} \in \mathbb{R}^n$$

$$\Rightarrow \exists\, \boldsymbol{q}_i \ni \boldsymbol{V}\boldsymbol{P}_{\boldsymbol{X}}\boldsymbol{x}_i = \boldsymbol{X}\boldsymbol{q}_i \quad \forall\, i = 1, \ldots, p,$$

where $\boldsymbol{x}_i$ denotes the i^{th} column of $\boldsymbol{X}$.

$$\Rightarrow \boldsymbol{V}\boldsymbol{P}_{\boldsymbol{X}}[\boldsymbol{x}_1, \ldots, \boldsymbol{x}_p] = \boldsymbol{X}[\boldsymbol{q}_1, \ldots, \boldsymbol{q}_p]$$

$$\Rightarrow \boldsymbol{V}\boldsymbol{P}_{\boldsymbol{X}}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{Q}, \text{ where } \boldsymbol{Q} = [\boldsymbol{q}_1, \ldots, \boldsymbol{q}_p]$$

$$\Rightarrow \boldsymbol{V}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{Q} \quad \text{for} \quad \boldsymbol{Q} = [\boldsymbol{q}_1, \ldots, \boldsymbol{q}_p].$$

□

Show that $\exists\, \boldsymbol{Q} \ni \boldsymbol{V}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{Q}$ in our previous example.

$$
\boldsymbol{V}\boldsymbol{X} = \begin{bmatrix} \boldsymbol{A} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{A} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{I} \\ \boldsymbol{I} \\ \boldsymbol{I} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{A} \\ \boldsymbol{A} \\ \boldsymbol{A} \end{bmatrix}
$$

$$
= \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{I} \\ \boldsymbol{I} \\ \boldsymbol{I} \end{bmatrix} \boldsymbol{A} = \boldsymbol{X}\boldsymbol{A} = \boldsymbol{X}\boldsymbol{Q}, \quad \text{where} \quad \boldsymbol{Q} = \boldsymbol{A}.
$$