The Aitken Model

The Aitken Model (AM):

Suppose

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon},$$

where

$$E(\boldsymbol{\varepsilon}) = \mathbf{0} \quad \text{and} \quad \text{Var}(\boldsymbol{\varepsilon}) = \sigma^2 \boldsymbol{V}$$

for some $\sigma^2 > 0$ and some known positive definite matrix $\boldsymbol{V}$.

Because $\sigma^2 \boldsymbol{V}$ is a variance matrix, $\boldsymbol{V}$ is symmetric and positive definite,
$\therefore \exists$ a symmetric and positive definite matrix $\boldsymbol{V}^{1/2} \ni$

$$\boldsymbol{V}^{1/2}\boldsymbol{V}^{1/2} = \boldsymbol{V} \quad \text{and} \quad \boldsymbol{V}^{1/2} \text{ is nonsingular with } \boldsymbol{V}^{-1/2} \equiv (\boldsymbol{V}^{1/2})^{-1}.$$

It follows that under the AM,

$$\boldsymbol{V}^{-1/2}\boldsymbol{y} = \boldsymbol{V}^{-1/2}\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{V}^{-1/2}\boldsymbol{\varepsilon} \iff \boldsymbol{z} = \boldsymbol{U}\boldsymbol{\beta} + \boldsymbol{\delta},$$

where

$$\boldsymbol{z} = \boldsymbol{V}^{-1/2}\boldsymbol{y}, \quad \boldsymbol{U} = \boldsymbol{V}^{-1/2}\boldsymbol{X}, \quad \text{and} \quad \boldsymbol{\delta} = \boldsymbol{V}^{-1/2}\boldsymbol{\varepsilon}$$

with

$$E(\boldsymbol{\delta}) = \boldsymbol{0}$$

and

$$\begin{aligned} \text{Var}(\boldsymbol{\delta}) &= \boldsymbol{V}^{-1/2}\sigma^2\boldsymbol{V}\boldsymbol{V}^{-1/2} \\ &= \sigma^2\boldsymbol{V}^{-1/2}\boldsymbol{V}^{1/2}\boldsymbol{V}^{1/2}\boldsymbol{V}^{-1/2} \\ &= \sigma^2\boldsymbol{I}. \end{aligned}$$

Thus, the AM for $\boldsymbol{y}$ is equivalent to the GMM for $\boldsymbol{z} = \boldsymbol{V}^{-1/2}\boldsymbol{y}$.

Estimability in the AM:

The AM is just a special case of the GLM.

Thus, as before, $\boldsymbol{c}'\boldsymbol{\beta}$ is estimable iff $\boldsymbol{c} \in \mathcal{C}(\boldsymbol{X}')$.

Note that

$$\begin{aligned}\mathcal{C}(\boldsymbol{X}') &= \boldsymbol{C}(\boldsymbol{X}'\boldsymbol{V}^{-1/2}) \\ &= \mathcal{C}((\boldsymbol{V}^{-1/2}\boldsymbol{X})') \\ &= \mathcal{C}(\boldsymbol{U}').\end{aligned}$$

Thus, $\boldsymbol{c} \in \mathcal{C}(\boldsymbol{X}') \iff \boldsymbol{c} \in \mathcal{C}(\boldsymbol{U}')$.

Let $\mathcal{L}_{\boldsymbol{y}}$ be the collection of all linear estimators that are linear in $\boldsymbol{y}$.

Let $\mathcal{L}_{\boldsymbol{z}}$ be the collection of all linear estimators in $\boldsymbol{z} = \boldsymbol{V}^{-1/2}\boldsymbol{y}$. Show that

$$\mathcal{L}_{\boldsymbol{y}} = \mathcal{L}_{\boldsymbol{z}}.$$

Proof:

Let $d + \boldsymbol{a}'\boldsymbol{y}$ be any arbitrary linear estimator in $\mathcal{L}_{\boldsymbol{y}}$. Then

$$\begin{aligned} d + \boldsymbol{a}'\boldsymbol{y} &= d + \boldsymbol{a}'\boldsymbol{V}^{1/2}\boldsymbol{V}^{-1/2}\boldsymbol{y} \\ &= d + \boldsymbol{a}'\boldsymbol{V}^{1/2}\boldsymbol{z} \\ &= d + \boldsymbol{h}'\boldsymbol{z} \in \mathcal{L}_{\boldsymbol{z}}, \quad \text{where} \quad \boldsymbol{h}' = \boldsymbol{a}'\boldsymbol{V}^{1/2}. \end{aligned}$$

Thus, $\mathcal{L}_{\boldsymbol{y}} \subseteq \mathcal{L}_{\boldsymbol{z}}$.

Now suppose $d + \boldsymbol{a}'\boldsymbol{z}$ is an arbitrary linear estimator in $\mathcal{L}_{\boldsymbol{z}}$. Then

$$\begin{aligned} d + \boldsymbol{a}'\boldsymbol{z} &= d + \boldsymbol{a}'\boldsymbol{V}^{-1/2}\boldsymbol{y} \\ &= d + \boldsymbol{h}'\boldsymbol{y} \in \mathcal{L}_{\boldsymbol{y}}, \quad \text{where} \quad \boldsymbol{h}' = \boldsymbol{a}'\boldsymbol{V}^{-1/2}. \end{aligned}$$

$\therefore \mathcal{L}_{\boldsymbol{z}} \subseteq \mathcal{L}_{\boldsymbol{y}}$ and it follows that $\mathcal{L}_{\boldsymbol{y}} = \mathcal{L}_{\boldsymbol{z}}$. $\square$

Estimating $E(\boldsymbol{y})$ under the Aitken Model:

Consider $Q_{\text{GLS}}(\boldsymbol{b}) = (\boldsymbol{y} - \boldsymbol{X}\boldsymbol{b})'\boldsymbol{V}^{-1}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{b})$.

Finding $\hat{\boldsymbol{\beta}}_{\text{GLS}}$ that minimizes $Q_{\text{GLS}}(\boldsymbol{b})$ over $\boldsymbol{b} \in \mathbb{R}^p$ is a Generalized Least Squares (GLS) problem.

If

$$Q_{\text{GLS}}(\hat{\boldsymbol{\beta}}_{\text{GLS}}) \leq Q_{\text{GLS}}(\boldsymbol{b}) \quad \forall\, \boldsymbol{b} \in \mathbb{R}^p,$$

$\hat{\boldsymbol{\beta}}_{\text{GLS}}$ is a solution to the GLS problem.

$\boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}}$ is known as GLS estimator of $E(\boldsymbol{y})$ if $\hat{\boldsymbol{\beta}}_{\text{GLS}}$ is a solution to the GLS problem.

Show that $\hat{\boldsymbol{\beta}}_{\text{GLS}}$ minimizes $Q_{\text{GLS}}(\boldsymbol{b})$ over $\boldsymbol{b} \in \mathbb{R}^p$ iff $\hat{\boldsymbol{\beta}}_{\text{GLS}}$ solves

$$\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X}\boldsymbol{b} = \boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y}.$$

These equations are known as the Aitken Equations (AE).

Proof:

$$
\begin{aligned}
&(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{b})'\boldsymbol{V}^{-1}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{b}) \\
&= (\boldsymbol{y} - \boldsymbol{X}\boldsymbol{b})'\boldsymbol{V}^{-1/2}\boldsymbol{V}^{-1/2}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{b}) \\
&= (\boldsymbol{V}^{-1/2}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{b}))'(\boldsymbol{V}^{-1/2}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{b})) \\
&= (\boldsymbol{V}^{-1/2}\boldsymbol{y} - \boldsymbol{V}^{-1/2}\boldsymbol{X}\boldsymbol{b})'(\boldsymbol{V}^{-1/2}\boldsymbol{y} - \boldsymbol{V}^{-1/2}\boldsymbol{X}\boldsymbol{b}) \\
&= (\boldsymbol{z} - \boldsymbol{U}\boldsymbol{b})'(\boldsymbol{z} - \boldsymbol{U}\boldsymbol{b}).
\end{aligned}
$$

By Result 2.3, $(\boldsymbol{z} - \boldsymbol{U}\boldsymbol{b})'(\boldsymbol{z} - \boldsymbol{U}\boldsymbol{b})$ is minimized at $\boldsymbol{b}^*$ iff $\boldsymbol{b}^*$ solves NE

$$\boldsymbol{U}'\boldsymbol{U}\boldsymbol{b} = \boldsymbol{U}'\boldsymbol{z}.$$

Now $\boldsymbol{U}'\boldsymbol{U}\boldsymbol{b} = \boldsymbol{U}'\boldsymbol{z}$ is equivalent to

$$
\begin{aligned}
&(\boldsymbol{V}^{-1/2}\boldsymbol{X})'(\boldsymbol{V}^{-1/2}\boldsymbol{X})\boldsymbol{b} = (\boldsymbol{V}^{-1/2}\boldsymbol{X})'(\boldsymbol{V}^{-1/2}\boldsymbol{y}) \\
&\iff \boldsymbol{X}'\boldsymbol{V}^{-1/2}\boldsymbol{V}^{-1/2}\boldsymbol{X}\boldsymbol{b} = \boldsymbol{X}'\boldsymbol{V}^{-1/2}\boldsymbol{V}^{-1/2}\boldsymbol{y} \\
&\iff \boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X}\boldsymbol{b} = \boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y}.
\end{aligned}
$$

$\therefore (\boldsymbol{y} - \boldsymbol{X}\boldsymbol{b})'\boldsymbol{V}^{-1}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{b})$ is minimized over $\boldsymbol{b} \in \mathbb{R}^p$ by $\boldsymbol{b}^*$ iff $\boldsymbol{b}^*$ solves the AE

$$\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X}\boldsymbol{b} = \boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y}.$$

$\square$

Henceforth, we will use $\hat{\boldsymbol{\beta}}_{\text{GLS}}$ to denote a solution to the AE.

We will use $\hat{\boldsymbol{\beta}}$ or $\hat{\boldsymbol{\beta}}_{\text{OLS}}$ to denote a solution to the NE

$$\boldsymbol{X}'\boldsymbol{X}\boldsymbol{b} = \boldsymbol{X}'\boldsymbol{y}. \quad \text{(\underline{O}rdinary \underline{L}east \underline{S}quares)}$$

Because of the equivalence between the AE

$$\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X}\boldsymbol{b} = \boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y}$$

and the NE

$$\boldsymbol{U}'\boldsymbol{U}\boldsymbol{b} = \boldsymbol{U}'\boldsymbol{z},$$

we know a solution to AE is

$$(\boldsymbol{U}'\boldsymbol{U})^{-}\boldsymbol{U}'\boldsymbol{z} = (\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y}.$$

Theorem 4.2 (Aitken Theorem):

Suppose the Aitken Model holds. If $\boldsymbol{c}'\boldsymbol{\beta}$ is estimable, then $\boldsymbol{c}'\hat{\boldsymbol{\beta}}_{\text{GLS}}$ is the BLUE of $\boldsymbol{c}'\boldsymbol{\beta}$.

Proof:

By Theorem 4.1, the BLUE of $\boldsymbol{c}'\boldsymbol{\beta}$ is

$$\boldsymbol{c}'(\boldsymbol{U}'\boldsymbol{U})^-\boldsymbol{U}'\boldsymbol{z} = \boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y}$$
$$= \boldsymbol{c}'\hat{\boldsymbol{\beta}}_{\text{GLS}}.$$

□

See also Exercises 4.22, 4.23.

Suppose $\boldsymbol{c}'\boldsymbol{\beta}$ is estimable.

Suppose the AM holds.

Find $\text{Var}(\boldsymbol{c}'\hat{\boldsymbol{\beta}}_{\text{GLS}})$.

We know $\boldsymbol{c}'\boldsymbol{\beta}$ is estimable under the AM

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

if and only if $\boldsymbol{c}'\boldsymbol{\beta}$ is estimable under the GMM

$$\boldsymbol{z} = \boldsymbol{U}\boldsymbol{\beta} + \boldsymbol{\delta}.$$

Furthermore, we know

$$\boldsymbol{c}'\hat{\boldsymbol{\beta}}_{\text{GLS}} = \boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y} = \boldsymbol{c}'(\boldsymbol{U}'\boldsymbol{U})^{-}\boldsymbol{U}'\boldsymbol{z}.$$

Thus,

$$
\begin{aligned}
\mathrm{Var}(\boldsymbol{c}'\hat{\boldsymbol{\beta}}_{\mathrm{GLS}}) &= \mathrm{Var}(\boldsymbol{c}'(\boldsymbol{U}'\boldsymbol{U})^{-}\boldsymbol{U}'\boldsymbol{z}) = \sigma^2\boldsymbol{c}'(\boldsymbol{U}'\boldsymbol{U})^{-}\boldsymbol{c} \\
&= \sigma^2\boldsymbol{c}'((\boldsymbol{V}^{-1/2}\boldsymbol{X})'(\boldsymbol{V}^{-1/2}\boldsymbol{X}))^{-}\boldsymbol{c} \\
&= \sigma^2\boldsymbol{c}'(\boldsymbol{X}'(\boldsymbol{V}^{-1/2})'\boldsymbol{V}^{-1/2}\boldsymbol{X})^{-}\boldsymbol{c} \\
&= \sigma^2\boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{V}^{-1/2}\boldsymbol{V}^{-1/2}\boldsymbol{X})^{-}\boldsymbol{c} \\
&= \sigma^2\boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-}\boldsymbol{c}.
\end{aligned}
$$

Estimation of σ^2 under the Aitken Model:

An unbiased estimator of σ^2 is $\frac{\boldsymbol{z}'(\boldsymbol{I}-\boldsymbol{P_U})\boldsymbol{z}}{n-r}$ based on our previous result for the GMM.

Now, note that

$$
\begin{aligned}
\boldsymbol{z}'(\boldsymbol{I}-\boldsymbol{P_U})\boldsymbol{z} &= \boldsymbol{z}'(\boldsymbol{I}-\boldsymbol{P_U})'(\boldsymbol{I}-\boldsymbol{P_U})\boldsymbol{z} \\
&= \|(\boldsymbol{I}-\boldsymbol{P_U})\boldsymbol{z}\|^2 = \|\boldsymbol{z}-\boldsymbol{P_U}\boldsymbol{z}\|^2 \\
&= \|\boldsymbol{z}-\boldsymbol{U}(\boldsymbol{U}'\boldsymbol{U})^-\boldsymbol{U}'\boldsymbol{z}\|^2 \\
&= \|\boldsymbol{z}-\boldsymbol{U}\hat{\boldsymbol{\beta}}_{\text{GLS}}\|^2 = \|\boldsymbol{V}^{-1/2}\boldsymbol{y}-\boldsymbol{V}^{-1/2}\boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}}\|^2 \\
&= \|\boldsymbol{V}^{-1/2}(\boldsymbol{y}-\boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}})\|^2 \\
&= (\boldsymbol{y}-\boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}})'\boldsymbol{V}^{-1}(\boldsymbol{y}-\boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}}).
\end{aligned}
$$

Thus,

$$\hat{\sigma}^2_{\text{GLS}} \equiv \frac{(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}})'\boldsymbol{V}^{-1}(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}})}{n - r}$$

is an unbiased estimator of σ^2 under the AM.

A Simple Example

Suppose for $i = 1, \ldots, n$,

$$y_i = \beta x_i + \varepsilon_i,$$

where $\varepsilon_1, \ldots, \varepsilon_n$ are uncorrelated, $E(\varepsilon_i) = 0$ and $\mathrm{Var}(\varepsilon_i) = \sigma^2 x_i > 0$.

Find the BLUE of β and an unbiased estimator of σ^2.

We have

$$\boldsymbol{X} = \boldsymbol{x}, \quad \boldsymbol{V} = diag(\boldsymbol{x}) = \begin{bmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & x_n \end{bmatrix}.$$

Then

$$\begin{aligned} \boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X} &= \boldsymbol{x}' diag(1/\boldsymbol{x})\boldsymbol{x} \\ &= \boldsymbol{1}'\boldsymbol{x} \\ &= \sum_{i=1}^{n} x_i. \end{aligned}$$

$$
\begin{aligned}
\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y} &= \boldsymbol{x}'diag(1/\boldsymbol{x})\boldsymbol{y} \\
&= \boldsymbol{1}'\boldsymbol{y} \\
&= \sum_{i=1}^{n} y_i. \\
\therefore \hat{\boldsymbol{\beta}}_{\text{GLS}} &= (\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}\boldsymbol{V}^{-1}\boldsymbol{y} \\
&= \frac{\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i}
\end{aligned}
$$

is the BLUE of $\boldsymbol{\beta}$.

Note that to find $\hat{\boldsymbol{\beta}}_{\text{GLS}}$ in this simple example, we solve a weighted least squares problem; i.e., $\hat{\boldsymbol{\beta}}_{\text{GLS}}$ minimizes

$$\begin{aligned} Q_{\text{GLS}}(\boldsymbol{b}) &= (\boldsymbol{y} - \boldsymbol{X}\boldsymbol{b})'\boldsymbol{V}^{-1}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{b}) \\ &= (\boldsymbol{y} - \boldsymbol{x}b)'diag(1/\boldsymbol{x})(\boldsymbol{y} - \boldsymbol{x}b) \\ &= \sum_{i=1}^{n} \frac{1}{x_i}(y_i - bx_i)^2. \end{aligned}$$

The weights in this case are $1/x_i$ $(i = 1, \ldots, n)$. Thus, the estimator pays more attention to $(y_i - bx_i)^2$ when x_i is small.

$$\hat{\sigma}^2_{\text{GLS}} = \frac{(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}})'\boldsymbol{V}^{-1}(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}})}{n-r}$$

$$= \frac{(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}})' diag(1/\boldsymbol{x})(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}_{\text{GLS}})}{n-r}$$

$$= \frac{\sum_{i=1}^{n} \frac{1}{x_i}(y_i - x_i \frac{\bar{y}}{\bar{x}})^2}{n-r}.$$

Find $\mathrm{Var}(\hat{\beta}_{\mathrm{GLS}})$ for this example.

$$
\begin{aligned}
\mathrm{Var}(\hat{\beta}_{\mathrm{GLS}}) &= \sigma^2 \boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-}\boldsymbol{c} \\
&= \sigma^2 1(\boldsymbol{x}' diag(1/\boldsymbol{x})\boldsymbol{x})^{-}1 \\
&= \sigma^2 \left(\sum_{i=1}^{n} x_i\right)^{-1} \\
&= \frac{\sigma^2}{\sum_{i=1}^{n} x_i}.
\end{aligned}
$$

Alternatively,

$$
\begin{aligned}
\text{Var}(\hat{\beta}_{\text{GLS}}) &= \text{Var}\left(\frac{\sum_{i=1}^n y_i}{\sum_{i=1}^n x_i}\right) \\
&= \frac{1}{(\sum_{i=1}^n x_i)^2} \text{Var}\left(\sum_{i=1}^n y_i\right) \\
&= \frac{1}{(\sum_{i=1}^n x_i)^2} \sum_{i=1}^n \text{Var}(y_i) \\
&= \frac{\sum_{i=1}^n \sigma^2 x_i}{(\sum_{i=1}^n x_i)^2} \\
&= \sigma^2 \frac{\sum_{i=1}^n x_i}{(\sum_{i=1}^n x_i)^2} \\
&= \frac{\sigma^2}{\sum_{i=1}^n x_i}.
\end{aligned}
$$

Find $\hat{\boldsymbol{\beta}}_{\text{OLS}}$ for this example.

$$
\begin{aligned}
\hat{\boldsymbol{\beta}}_{\text{OLS}} &= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y} \\
&= (\boldsymbol{x}'\boldsymbol{x})^{-1}\boldsymbol{x}'\boldsymbol{y} \\
&= \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}.
\end{aligned}
$$

Find $\mathrm{Var}(\hat{\beta}_{\mathrm{OLS}})$ in this example.

$$
\begin{aligned}
\text{Var}(\hat{\boldsymbol{\beta}}_{\text{OLS}}) &= \text{Var}((\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}) \\
&= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'(\sigma^2\boldsymbol{V})\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1} \\
&= \sigma^2(\boldsymbol{x}'\boldsymbol{x})^{-1}\boldsymbol{x}'diag(\boldsymbol{x})\boldsymbol{x}(\boldsymbol{x}'\boldsymbol{x})^{-1} \\
&= \sigma^2\frac{\sum_{i=1}^n x_i^3}{(\sum_{i=1}^n x_i^2)^2}.
\end{aligned}
$$

Alternatively,

$$\begin{aligned}\mathrm{Var}(\hat{\beta}_{\mathrm{OLS}}) &= \mathrm{Var}\left(\frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2}\right) \\ &= \frac{1}{(\sum_{i=1}^n x_i^2)^2}\sum_{i=1}^n x_i^2 \mathrm{Var}(y_i) \\ &= \sigma^2 \frac{\sum_{i=1}^n x_i^3}{(\sum_{i=1}^n x_i^2)^2}.\end{aligned}$$

$$\text{Var}(\hat{\beta}_{\text{OLS}}) = \sigma^2 \frac{\sum_{i=1}^n x_i^3}{(\sum_{i=1}^n x_i^2)^2}$$
$$\geq \frac{\sigma^2}{\sum_{i=1}^n x_i} = \text{Var}(\hat{\beta}_{\text{GLS}}),$$

with equality iff

$$x_1 = \cdots = x_n;$$

i.e., iff GMM holds.