The Trace of a Matrix

The trace of a square matrix $\underset{n\times n}{\boldsymbol{A}} = [a_{ij}]$ is

$$trace(\boldsymbol{A}) = tr(\boldsymbol{A}) = \sum_{i=1}^{n} a_{ii}.$$

For example,

$$tr\left(\begin{bmatrix} 5 & 3 & 5 \\ 4 & -1 & 2 \\ -3 & 8 & 7 \end{bmatrix}\right) = 5 - 1 + 7 = 11.$$

Some Simple Facts about Trace

Suppose $k, k_1, \ldots, k_m \in \mathbb{R}$ and $\boldsymbol{A}, \boldsymbol{A}_1, \ldots, \boldsymbol{A}_m$ are each $n \times n$ matrices.

Then

1. $tr(\boldsymbol{A}) = tr(\boldsymbol{A}')$
2. $tr(k\boldsymbol{A}) = k \cdot tr(\boldsymbol{A})$
3. $tr(\boldsymbol{A}_1 + \boldsymbol{A}_2) = tr(\boldsymbol{A}_1) + tr(\boldsymbol{A}_2)$
4. $tr(\sum_{i=1}^m k_i \boldsymbol{A}_i) = \sum_{i=1}^m k_i \cdot tr(\boldsymbol{A}_i)$

Result A.17:

(a) $tr(\boldsymbol{AB}) = tr(\boldsymbol{BA})$. This is known as the cyclic property of the trace.

(b) If $\underset{m\times n}{\boldsymbol{A}} = [a_{ij}]$, then

$$tr(\boldsymbol{A}'\boldsymbol{A}) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2.$$

Proof of Result A.17: HW problem. □

Suppose $\boldsymbol{A}$ is an $m \times n$ matrix of rank r. Prove that there exist matrices $\underset{m \times r}{\boldsymbol{B}}$ and $\underset{r \times n}{\boldsymbol{C}}$ such that

$$\boldsymbol{A} = \boldsymbol{BC} \text{ and } rank(\boldsymbol{B}) = rank(\boldsymbol{C}) = r.$$

Suppose $\boldsymbol{A}$ is an $n \times n$ matrix such that $\boldsymbol{AA} = k\boldsymbol{A}$ for some $k \in \mathbb{R}$. Prove that $tr(\boldsymbol{A}) = k \cdot rank(\boldsymbol{A})$.

(Note that this result implies the trace of an idempotent matrix is equal to its rank.)

Prove that $tr(\boldsymbol{I} - \boldsymbol{P_X}) = n - rank(\boldsymbol{X})$.