Constraints on the Parameter Vector

Suppose

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon},$$

where $E(\boldsymbol{\varepsilon}) = \mathbf{0}$ and $\boldsymbol{\beta} \in \mathbb{R}^p$ satisfies $\boldsymbol{H}'\boldsymbol{\beta} = \boldsymbol{h}$ for some known $\underset{p\times q}{\boldsymbol{H}}$ of rank q and some known $\underset{q\times 1}{\boldsymbol{h}}$.

Given that we know $\boldsymbol{\beta}$ satisfies $\boldsymbol{H}'\boldsymbol{\beta} = \boldsymbol{h}$, what functions $\boldsymbol{c}'\boldsymbol{\beta}$ are estimable, and how do we estimate them?

A linear estimator $d + \boldsymbol{a}'\boldsymbol{y}$ is unbiased for $\boldsymbol{c}'\boldsymbol{\beta}$ in the restricted model iff

$$E(d + \boldsymbol{a}'\boldsymbol{y}) = \boldsymbol{c}'\boldsymbol{\beta} \quad \forall \; \boldsymbol{\beta} \ni \boldsymbol{H}'\boldsymbol{\beta} = \boldsymbol{h}.$$

$\boldsymbol{c}'\boldsymbol{\beta}$ is estimable in the restricted model iff $\exists$ a linear estimator $d + \boldsymbol{a}'\boldsymbol{y} \ni$

$$E(d + \boldsymbol{a}'\boldsymbol{y}) = \boldsymbol{c}'\boldsymbol{\beta} \quad \forall\ \boldsymbol{\beta} \text{ satisfying } \boldsymbol{H}'\boldsymbol{\beta} = \boldsymbol{h},$$

i.e., iff $\exists$ a linear estimator that is unbiased for $\boldsymbol{c}'\boldsymbol{\beta}$ in the restricted model.

Result 3.7:

In the restricted model, $d + \boldsymbol{a}'\boldsymbol{y}$ is unbiased for $\boldsymbol{c}'\boldsymbol{\beta}$ iff

$$\exists\, \boldsymbol{l} \ni \boldsymbol{c} = \boldsymbol{X}'\boldsymbol{a} + \boldsymbol{H}\boldsymbol{l} \quad \text{and} \quad d = \boldsymbol{l}'\boldsymbol{h}.$$

Proof of Result 3.7:

($\Longleftarrow$) Suppose

$$\exists\, \boldsymbol{l} \ni \boldsymbol{c} = \boldsymbol{X}'\boldsymbol{a} + \boldsymbol{H}\boldsymbol{l} \quad \text{and} \quad d = \boldsymbol{l}'\boldsymbol{h}.$$

Then

$$\begin{aligned} E(d + \boldsymbol{a}'\boldsymbol{y}) &= \boldsymbol{l}'\boldsymbol{h} + \boldsymbol{a}'\boldsymbol{X}\boldsymbol{\beta} \\ &= \boldsymbol{l}'\boldsymbol{H}'\boldsymbol{\beta} + \boldsymbol{a}'\boldsymbol{X}\boldsymbol{\beta} \quad \forall\, \boldsymbol{\beta} \ni \boldsymbol{H}'\boldsymbol{\beta} = \boldsymbol{h} \\ &= (\boldsymbol{l}'\boldsymbol{H}' + \boldsymbol{a}'\boldsymbol{X})\boldsymbol{\beta} \quad \forall\, \boldsymbol{\beta} \ni \boldsymbol{H}'\boldsymbol{\beta} = \boldsymbol{h} \\ &= (\boldsymbol{X}'\boldsymbol{a} + \boldsymbol{H}\boldsymbol{l})'\boldsymbol{\beta} \quad \forall\, \boldsymbol{\beta} \ni \boldsymbol{H}'\boldsymbol{\beta} = \boldsymbol{h} \\ &= \boldsymbol{c}'\boldsymbol{\beta} \quad \forall\, \boldsymbol{\beta} \ni \boldsymbol{H}'\boldsymbol{\beta} = \boldsymbol{h}. \end{aligned}$$

$(\Longrightarrow)$ First note that

$$\{\boldsymbol{\beta} : \boldsymbol{H}'\boldsymbol{\beta} = \boldsymbol{h}\} = \{(\boldsymbol{H}')^{-}\boldsymbol{h} + (\boldsymbol{I} - (\boldsymbol{H}')^{-}\boldsymbol{H}')\boldsymbol{z} : \boldsymbol{z} \in \mathbb{R}^p\}$$
$$= \{\boldsymbol{b}^* + \boldsymbol{W}\boldsymbol{z} : \boldsymbol{z} \in \mathbb{R}^p\},$$

where $\boldsymbol{b}^* = (\boldsymbol{H}')^{-}\boldsymbol{h}$ is one particular solution to $\boldsymbol{H}'\boldsymbol{\beta} = \boldsymbol{h}$ and $\mathcal{C}(\boldsymbol{W}) = \mathcal{N}(\boldsymbol{H}')$ by Results A.12 and A.15, respectively.

Now suppose

$$E(d + \boldsymbol{a}'\boldsymbol{y}) = d + \boldsymbol{a}'\boldsymbol{X}\boldsymbol{\beta} = \boldsymbol{c}'\boldsymbol{\beta} \quad \forall\ \boldsymbol{\beta} \ni \boldsymbol{H}'\boldsymbol{\beta} = \boldsymbol{h}.$$

This is equivalent to

$$d + \boldsymbol{a}'\boldsymbol{X}(\boldsymbol{b}^* + \boldsymbol{W}\boldsymbol{z}) = \boldsymbol{c}'(\boldsymbol{b}^* + \boldsymbol{W}\boldsymbol{z}) \quad \forall\ \boldsymbol{z} \in \mathbb{R}^p$$

$$\iff d + \boldsymbol{a}'\boldsymbol{X}\boldsymbol{b}^* - \boldsymbol{c}'\boldsymbol{b}^* + (\boldsymbol{a}'\boldsymbol{X} - \boldsymbol{c}')\boldsymbol{W}\boldsymbol{z} = 0 \quad \forall\ \boldsymbol{z} \in \mathbb{R}^p$$

$$\iff d + \boldsymbol{a}'\boldsymbol{X}\boldsymbol{b}^* - \boldsymbol{c}'\boldsymbol{b}^* = 0 \quad \text{and} \quad \boldsymbol{W}'(\boldsymbol{X}'\boldsymbol{a} - \boldsymbol{c}) = \boldsymbol{0} \quad \text{by Result A.8.}$$

Now $\boldsymbol{W}'(\boldsymbol{X}'\boldsymbol{a} - \boldsymbol{c}) = \boldsymbol{0}$ implies that

$$\begin{aligned} \boldsymbol{X}'\boldsymbol{a} - \boldsymbol{c} \in \mathcal{N}(\boldsymbol{W}') &= \mathcal{C}(\boldsymbol{W})^{\perp} \\ &= \mathcal{N}(\boldsymbol{H}')^{\perp} \\ &= \mathcal{C}(\boldsymbol{H}). \end{aligned}$$

$\therefore \exists\, \boldsymbol{m} \ni \boldsymbol{H}\boldsymbol{m} = \boldsymbol{X}'\boldsymbol{a} - \boldsymbol{c}$

$\Rightarrow \exists\, \boldsymbol{m} \ni \boldsymbol{c} = \boldsymbol{X}'\boldsymbol{a} - \boldsymbol{H}\boldsymbol{m}$

$\Rightarrow \exists\, \boldsymbol{l} \ni \boldsymbol{c} = \boldsymbol{X}'\boldsymbol{a} + \boldsymbol{H}\boldsymbol{l}. \quad (\boldsymbol{l} = -\boldsymbol{m}.)$

Now

$$
\begin{aligned}
d + \boldsymbol{a}'\boldsymbol{X}\boldsymbol{b}^* - \boldsymbol{c}'\boldsymbol{b}^* = 0 \Rightarrow d &= \boldsymbol{c}'\boldsymbol{b}^* - \boldsymbol{a}'\boldsymbol{X}\boldsymbol{b}^* \\
&= (\boldsymbol{X}'\boldsymbol{a} + \boldsymbol{H}\boldsymbol{l})'\boldsymbol{b}^* - \boldsymbol{a}'\boldsymbol{X}\boldsymbol{b}^* \\
&= \boldsymbol{l}'\boldsymbol{H}'\boldsymbol{b}^* + \boldsymbol{a}'\boldsymbol{X}\boldsymbol{b}^* - \boldsymbol{a}'\boldsymbol{X}\boldsymbol{b}^* \\
&= \boldsymbol{l}'\boldsymbol{H}'\boldsymbol{b}^* \\
&= \boldsymbol{l}'\boldsymbol{h}.
\end{aligned}
$$

$\square$

Recall that in the unrestricted case, $\boldsymbol{c}'\boldsymbol{\beta}$ is estimable iff $\boldsymbol{c} \in \mathcal{C}(\boldsymbol{X}')$.

Result 3.7 says that $\boldsymbol{c}'\boldsymbol{\beta}$ is estimable in the restricted case iff $\boldsymbol{c} \in \mathcal{C}([\boldsymbol{X}', \boldsymbol{H}])$.

Thus $\boldsymbol{c}'\boldsymbol{\beta}$ is estimable under unrestricted model $\Rightarrow \boldsymbol{c}'\boldsymbol{\beta}$ estimable under restricted model.

However, the converse doesn't hold.

If $\mathcal{C}(\boldsymbol{X}') \subset \mathcal{C}([\boldsymbol{X}', \boldsymbol{H}]), \exists$ functions $\boldsymbol{c}'\boldsymbol{\beta}$ estimable in restricted case but nonestimable in unrestricted case.

Example:

Consider the one-way ANOVA model

$$E(\boldsymbol{y}_{ij}) = \mu + \tau_i \quad i = 1, \ldots, t \quad \text{and} \quad j = 1, \ldots, n_i.$$

Show that $\boldsymbol{c}'\boldsymbol{\beta}$ is estimable $\forall\, \boldsymbol{c} \in \mathbb{R}^p$ under restriction

$$\tau_1 + \cdots + \tau_t = 0.$$

Now suppose we consider the constraints

$$\tau_1 = \tau_2 = \cdots = \tau_t.$$

What functions $\boldsymbol{c}'\boldsymbol{\beta}$ are estimable in this case?

The Restricted Normal Equations (RNE) are

$$\begin{bmatrix} \boldsymbol{X'X} & \boldsymbol{H} \\ \boldsymbol{H'} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X'y} \\ \boldsymbol{h} \end{bmatrix}.$$

Result 3.8:

The RNE are consistent.

Proof: First show
$$\begin{bmatrix} \boldsymbol{X}'\boldsymbol{y} \\ \boldsymbol{h} \end{bmatrix} \in \mathcal{C}\left(\begin{bmatrix} \boldsymbol{X}' & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{H}' \end{bmatrix}\right).$$

Now suppose that we can show

$$\mathcal{N}\left(\begin{bmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{H} \\ \boldsymbol{H}' & \boldsymbol{0} \end{bmatrix}\right) \subseteq \mathcal{N}\left(\begin{bmatrix} \boldsymbol{X} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{H} \end{bmatrix}\right).$$

Explain why this implies the RNE are consistent.

Now show that

$$\mathcal{N}\left(\begin{bmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{H} \\ \boldsymbol{H}' & \boldsymbol{0} \end{bmatrix}\right) \subseteq \mathcal{N}\left(\begin{bmatrix} \boldsymbol{X} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{H} \end{bmatrix}\right).$$

Result 3.9:

If $\tilde{\beta}$ is the first p components of a solution to the RNE, then $\tilde{\beta}$ minimizes

$$Q(\boldsymbol{b}) = (\boldsymbol{y} - \boldsymbol{X}\boldsymbol{b})'(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{b}) = \|\boldsymbol{y} - \boldsymbol{X}\boldsymbol{b}\|^2$$

over $\boldsymbol{b}$ satisfying $\boldsymbol{H}'\boldsymbol{b} = \boldsymbol{h}$.

Result 3.10:

If $\tilde{\boldsymbol{\beta}}$ satisfies

$$\boldsymbol{H}'\tilde{\boldsymbol{\beta}} = \boldsymbol{h} \quad \text{and} \quad Q(\tilde{\boldsymbol{\beta}}) \leq Q(\boldsymbol{b}) \quad \forall\, \boldsymbol{b} \ni \boldsymbol{H}'\boldsymbol{b} = \boldsymbol{h},$$

then $\tilde{\boldsymbol{\beta}}$ is the first p components of a solution to the RNE.