Estimable Functions and Their Least Squares Estimators in Reparameterized Models

Once again consider the linear models

$$\boldsymbol{y} = \boldsymbol{W}\boldsymbol{\alpha} + \boldsymbol{\varepsilon} \quad \text{and} \quad \boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon},$$

where $E(\boldsymbol{\varepsilon}) = \mathbf{0}$ and $\mathcal{C}(\boldsymbol{W}) = \mathcal{C}(\boldsymbol{X})$.

Suppose $\boldsymbol{W} = \boldsymbol{X}\boldsymbol{T}$ and $\boldsymbol{X} = \boldsymbol{W}\boldsymbol{S}$.

We have

$$\begin{aligned} E(\boldsymbol{y}) &= \boldsymbol{X\beta} = \boldsymbol{W\alpha} \\ &= \boldsymbol{WS\beta} = \boldsymbol{XT\alpha}. \end{aligned}$$

Note the correspondence between

$$\boldsymbol{\beta} \quad \text{and} \quad \boldsymbol{T\alpha}$$

$$\boldsymbol{\alpha} \quad \text{and} \quad \boldsymbol{S\beta}.$$

Result 3.4:

Suppose $\boldsymbol{c}'\boldsymbol{\beta}$ is estimable and $\boldsymbol{W}'\boldsymbol{W}\hat{\boldsymbol{\alpha}} = \boldsymbol{W}'\boldsymbol{y}$. Then $\boldsymbol{c}'\boldsymbol{T}\hat{\boldsymbol{\alpha}}$ is the least squares estimator of $\boldsymbol{c}'\boldsymbol{\beta}$.

Proof of Result 3.4:

$\boldsymbol{W}'\boldsymbol{W}\hat{\boldsymbol{\alpha}} = \boldsymbol{W}'\boldsymbol{y} \Rightarrow \boldsymbol{X}'\boldsymbol{X}\boldsymbol{T}\hat{\boldsymbol{\alpha}} = \boldsymbol{X}'\boldsymbol{y}$ by Result 2.9.

$\because \boldsymbol{T}\hat{\boldsymbol{\alpha}}$ solves the NE $\boldsymbol{X}'\boldsymbol{X}\boldsymbol{b} = \boldsymbol{X}'\boldsymbol{y}$,

$\boldsymbol{c}'\boldsymbol{T}\hat{\boldsymbol{\alpha}}$ is LS estimator of $\boldsymbol{c}'\boldsymbol{\beta}$ by definition. $\square$

Example:

Consider again the case where

$$\boldsymbol{X} = \begin{bmatrix} \mathbf{1}_{n_1} & \mathbf{1}_{n_1} & \mathbf{0}_{n_1} \\ \mathbf{1}_{n_2} & \mathbf{0}_{n_2} & \mathbf{1}_{n_2} \end{bmatrix} \quad \boldsymbol{T} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boldsymbol{W} = \begin{bmatrix} \mathbf{1}_{n_1} & \mathbf{0}_{n_1} \\ \mathbf{1}_{n_2} & \mathbf{1}_{n_2} \end{bmatrix} \qquad \boldsymbol{S} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Recall that the unique solution to the NE

$$\boldsymbol{W}'\boldsymbol{W}\hat{\boldsymbol{\alpha}} = \boldsymbol{W}'\boldsymbol{y}$$

is

$$\hat{\boldsymbol{\alpha}} = \begin{bmatrix} \bar{y}_{1\cdot} \\ \bar{y}_{2\cdot} - \bar{y}_{1\cdot} \end{bmatrix}.$$

Suppose we denote the components of $\boldsymbol{\beta}$ by μ, τ_1, τ_2 so that

$$\boldsymbol{\beta} = \begin{bmatrix} \mu \\ \tau_1 \\ \tau_2 \end{bmatrix} \quad \text{and} \quad E(\boldsymbol{y}) = \begin{bmatrix} (\mu + \tau_1)\mathbf{1}_{n_1} \\ (\mu + \tau_2)\mathbf{1}_{n_2} \end{bmatrix}.$$

$\tau_1 - \tau_2 = (\mu + \tau_1) - (\mu + \tau_2)$ is estimable $\because$ it is a LC of elements of $E(\boldsymbol{y})$.

$\tau_1 - \tau_2 = \boldsymbol{c}'\boldsymbol{\beta}$ where

$$\boldsymbol{c}' = [0, 1, -1].$$

Result 3.4 implies that

$$\boldsymbol{c}'\boldsymbol{T}\hat{\boldsymbol{\alpha}} = [0, 1, -1] \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{y}_{1\cdot} \\ \bar{y}_{2\cdot} - \bar{y}_{1\cdot} \end{bmatrix}$$

$$= [0, -1] \begin{bmatrix} \bar{y}_{1\cdot} \\ \bar{y}_{2\cdot} - \bar{y}_{1\cdot} \end{bmatrix}$$

$$= \bar{y}_{1\cdot} - \bar{y}_{2\cdot}$$

is LSE of $\boldsymbol{c}'\boldsymbol{\beta} = \tau_1 - \tau_2$.

Result 3.5:

If $\boldsymbol{d}'\boldsymbol{\alpha}$ is estimable in the model $\boldsymbol{y} = \boldsymbol{W}\boldsymbol{\alpha} + \boldsymbol{\varepsilon}$, then $\boldsymbol{d}'\boldsymbol{S}\boldsymbol{\beta}$ is estimable in the model $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, and its LSE is $\boldsymbol{d}'\hat{\boldsymbol{\alpha}} = \boldsymbol{d}'\boldsymbol{S}\hat{\boldsymbol{\beta}}$, where $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$ are solutions to

$$\boldsymbol{W}'\boldsymbol{W}\boldsymbol{a} = \boldsymbol{W}'\boldsymbol{y} \quad \text{and} \quad \boldsymbol{X}'\boldsymbol{X}\boldsymbol{b} = \boldsymbol{X}'\boldsymbol{y}, \quad \text{respectively.}$$

Proof of Result 3.5:

By Result 3.1, $\boldsymbol{d}'\boldsymbol{\alpha}$ is estimable $\iff \exists\ \boldsymbol{a} \ni \boldsymbol{d}' = \boldsymbol{a}'\boldsymbol{W}$.

Multiplying on the right by $\boldsymbol{S}$ leads to $\exists\ \boldsymbol{a} \ni \boldsymbol{d}'\boldsymbol{S} = \boldsymbol{a}'\boldsymbol{W}\boldsymbol{S} = \boldsymbol{a}'\boldsymbol{X}$.

$\therefore$ By Result 3.1, $\boldsymbol{d}'\boldsymbol{S}\boldsymbol{\beta}$ is estimable in the model $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$.

By definition, we know $\boldsymbol{d}'\hat{\boldsymbol{\alpha}}$ is LS estimate of $\boldsymbol{d}'\boldsymbol{\alpha}$ and $\boldsymbol{d}'\boldsymbol{S}\hat{\boldsymbol{\beta}}$ is LS estimate of $\boldsymbol{d}'\boldsymbol{S}\boldsymbol{\beta}$.

To see that $\boldsymbol{d}'\hat{\boldsymbol{\alpha}} = \boldsymbol{d}'\boldsymbol{S}\hat{\boldsymbol{\beta}}$, note that Result 3.4 implies

$$\begin{aligned} \boldsymbol{d}'\boldsymbol{S}\hat{\boldsymbol{\beta}} &= \boldsymbol{d}'\boldsymbol{S}\boldsymbol{T}\hat{\boldsymbol{\alpha}} \\ &= \boldsymbol{a}'\boldsymbol{W}\boldsymbol{S}\boldsymbol{T}\hat{\boldsymbol{\alpha}} \quad (\boldsymbol{d}' = \boldsymbol{a}'\boldsymbol{W}) \\ &= \boldsymbol{a}'\boldsymbol{X}\boldsymbol{T}\hat{\boldsymbol{\alpha}} \quad (\boldsymbol{X} = \boldsymbol{W}\boldsymbol{S}) \\ &= \boldsymbol{a}'\boldsymbol{W}\hat{\boldsymbol{\alpha}} \quad (\boldsymbol{W} = \boldsymbol{X}\boldsymbol{T}) \\ &= \boldsymbol{d}'\hat{\boldsymbol{\alpha}}. \end{aligned}$$

□

Returning to our example, $rank(\boldsymbol{W}) = 2 \Rightarrow \boldsymbol{d}'\boldsymbol{\alpha}$ is estimable $\forall\ \boldsymbol{d} \in \mathbb{R}^2$.

For example, $\boldsymbol{d}'\boldsymbol{\alpha}$ is estimable for $\boldsymbol{d}' = [1, 0]$, and the LSE is

$$\boldsymbol{d}'\hat{\boldsymbol{\alpha}} = [1, 0] \begin{bmatrix} \bar{y}_{1\cdot} \\ \bar{y}_{2\cdot} - \bar{y}_{1\cdot} \end{bmatrix} = \bar{y}_{1\cdot} \quad .$$

According to Result 3.5,

$$\boldsymbol{d}'\boldsymbol{S}\boldsymbol{\beta} = [1, 0] \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \tau_1 \\ \tau_2 \end{bmatrix}$$

$$= [1, 1, 0] \begin{bmatrix} \mu \\ \tau_1 \\ \tau_2 \end{bmatrix}$$

$$= \mu + \tau_1 \quad \text{is also estimable.}$$

The LSE is

$$\begin{aligned} \boldsymbol{d}'\boldsymbol{S}\hat{\boldsymbol{\beta}} &= [1, 0] \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \bar{y}_{1\cdot} \\ \bar{y}_{2\cdot} \end{bmatrix} \\ &= [1, 1, 0] \begin{bmatrix} 0 \\ \bar{y}_{1\cdot} \\ \bar{y}_{2\cdot} \end{bmatrix} \\ &= \bar{y}_{1\cdot} \quad . \end{aligned}$$