Equivalent Models and Reparameterization

Two linear models

$$\boldsymbol{y} = \boldsymbol{W\alpha} + \boldsymbol{\varepsilon} \quad \text{and} \quad \boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon}$$

are equivalent, or <u>reparameterizations</u> of each other, iff

$$\mathcal{C}(\boldsymbol{X}) = \mathcal{C}(\boldsymbol{W}).$$

Result 2.8:

If $\mathcal{C}(\boldsymbol{X}) = \mathcal{C}(\boldsymbol{W})$, then $\boldsymbol{P_X} = \boldsymbol{P_W}$.

Proof of Result 2.8:

$\forall\, \boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{y} = \boldsymbol{P_X y} + (\boldsymbol{I} - \boldsymbol{P_X})\boldsymbol{y} = \boldsymbol{P_W y} + (\boldsymbol{I} - \boldsymbol{P_W})\boldsymbol{y}$.

We have written $\boldsymbol{y}$ as a sum of a vector in $\mathcal{C}(\boldsymbol{X}) = \mathcal{C}(\boldsymbol{W})$ and a vector in $\mathcal{C}(\boldsymbol{X})^{\perp} = \mathcal{C}(\boldsymbol{W})^{\perp}$.

Such a decomposition is unique by Result A.4.

Therefore, $\boldsymbol{P_X y} = \boldsymbol{P_W y} \quad \forall\, \boldsymbol{y} \in \mathbb{R}^n \Rightarrow \boldsymbol{P_X} = \boldsymbol{P_W}$. □

Corollary 2.4:

If $\mathcal{C}(\boldsymbol{X}) = \mathcal{C}(\boldsymbol{W})$, then

$$\hat{\boldsymbol{y}} = \boldsymbol{P_X y} = \boldsymbol{P_W y} \quad \text{and}$$
$$\hat{\boldsymbol{\varepsilon}} = (\boldsymbol{I} - \boldsymbol{P_X})\boldsymbol{y} = (\boldsymbol{I} - \boldsymbol{P_W})\boldsymbol{y} = \boldsymbol{y} - \hat{\boldsymbol{y}}.$$

The fitted values and residuals are the same for the models that are reparameterizations of one another.

Recall $\mathcal{C}(\boldsymbol{X}) = \mathcal{C}(\boldsymbol{W}) \Longleftrightarrow$

$\boldsymbol{W} = \boldsymbol{X}\boldsymbol{T}$ for some matrix $\boldsymbol{T}$ and

$\boldsymbol{X} = \boldsymbol{W}\boldsymbol{S}$ for some matrix $\boldsymbol{S}$.

<u>Result 2.9:</u> If $\mathcal{C}(\boldsymbol{W}) = \mathcal{C}(\boldsymbol{X})$ and $\hat{\alpha}$ solves the NE $\boldsymbol{W}'\boldsymbol{W}\boldsymbol{a} = \boldsymbol{W}'\boldsymbol{y}$, then $\hat{\beta} = \boldsymbol{T}\hat{\alpha}$ solves the NE $\boldsymbol{X}'\boldsymbol{X}\boldsymbol{b} = \boldsymbol{X}'\boldsymbol{y}$, where $\boldsymbol{T}$ is defined by $\boldsymbol{W} = \boldsymbol{X}\boldsymbol{T}$.

Example:

Suppose

$$\boldsymbol{y} = \begin{bmatrix} y_{11} \\ y_{12} \\ \vdots \\ y_{1n_1} \\ y_{21} \\ y_{22} \\ \vdots \\ y_{2n_2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{bmatrix}.$$

Consider

$$\boldsymbol{X} = \begin{bmatrix} \mathbf{1}_{n_1} & \mathbf{1}_{n_1} & \mathbf{0}_{n_1} \\ \mathbf{1}_{n_2} & \mathbf{0}_{n_2} & \mathbf{1}_{n_2} \end{bmatrix} \quad \text{and} \quad \boldsymbol{W} = \begin{bmatrix} \mathbf{1}_{n_1} & \mathbf{0}_{n_1} \\ \mathbf{1}_{n_2} & \mathbf{1}_{n_2} \end{bmatrix}.$$

Find $\boldsymbol{T} \ni \boldsymbol{W} = \boldsymbol{X}\boldsymbol{T}$.

Find $\hat{\boldsymbol{\beta}}$ a solution to $\boldsymbol{X}'\boldsymbol{X}\boldsymbol{b} = \boldsymbol{X}'\boldsymbol{y}$.

Find $\hat{\boldsymbol{\alpha}}$ a solution to $\boldsymbol{W}'\boldsymbol{W}\boldsymbol{a} = \boldsymbol{W}'\boldsymbol{y}$.

Show that $\boldsymbol{X}\hat{\boldsymbol{\beta}} = \boldsymbol{W}\hat{\boldsymbol{\alpha}}$.