Equivalent Models and Reparameterization

Two linear models

$$\boldsymbol{y} = \boldsymbol{W}\boldsymbol{\alpha} + \boldsymbol{\varepsilon} \quad \text{and} \quad \boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

are equivalent, or reparameterizations of each other, iff

$$\mathcal{C}(\boldsymbol{X}) = \mathcal{C}(\boldsymbol{W}).$$

Result 2.8:

If $\mathcal{C}(\boldsymbol{X}) = \mathcal{C}(\boldsymbol{W})$, then $\boldsymbol{P_X} = \boldsymbol{P_W}$.

Proof of Result 2.8:

$\forall\, \boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{y} = \boldsymbol{P_X y} + (\boldsymbol{I} - \boldsymbol{P_X})\boldsymbol{y} = \boldsymbol{P_W y} + (\boldsymbol{I} - \boldsymbol{P_W})\boldsymbol{y}$.

We have written $\boldsymbol{y}$ as a sum of a vector in $\mathcal{C}(\boldsymbol{X}) = \mathcal{C}(\boldsymbol{W})$ and a vector in $\mathcal{C}(\boldsymbol{X})^\perp = \mathcal{C}(\boldsymbol{W})^\perp$.

Such a decomposition is unique by Result A.4.

Therefore, $\boldsymbol{P_X y} = \boldsymbol{P_W y} \quad \forall\, \boldsymbol{y} \in \mathbb{R}^n \Rightarrow \boldsymbol{P_X} = \boldsymbol{P_W}$. $\square$

Corollary 2.4:

If $\mathcal{C}(\boldsymbol{X}) = \mathcal{C}(\boldsymbol{W})$, then

$$\hat{\boldsymbol{y}} = \boldsymbol{P}_X \boldsymbol{y} = \boldsymbol{P}_W \boldsymbol{y} \quad \text{and}$$
$$\hat{\boldsymbol{\varepsilon}} = (\boldsymbol{I} - \boldsymbol{P}_X)\boldsymbol{y} = (\boldsymbol{I} - \boldsymbol{P}_W)\boldsymbol{y} = \boldsymbol{y} - \hat{\boldsymbol{y}}.$$

The fitted values and residuals are the same for the models that are reparameterizations of one another.

Recall $\mathcal{C}(\boldsymbol{X}) = \mathcal{C}(\boldsymbol{W}) \iff$

$$\boldsymbol{W} = \boldsymbol{XT} \text{ for some matrix } \boldsymbol{T} \text{ and}$$

$$\boldsymbol{X} = \boldsymbol{WS} \text{ for some matrix } \boldsymbol{S}.$$

Result 2.9: If $\mathcal{C}(\boldsymbol{W}) = \mathcal{C}(\boldsymbol{X})$ and $\hat{\alpha}$ solves the NE $\boldsymbol{W}'\boldsymbol{W}\boldsymbol{a} = \boldsymbol{W}'\boldsymbol{y}$, then $\hat{\boldsymbol{\beta}} = \boldsymbol{T}\hat{\alpha}$ solves the NE $\boldsymbol{X}'\boldsymbol{X}\boldsymbol{b} = \boldsymbol{X}'\boldsymbol{y}$, where $\boldsymbol{T}$ is defined by $\boldsymbol{W} = \boldsymbol{XT}$.

Proof of Results 2.9:

$\hat{\alpha}$ a solution to NE

$$\boldsymbol{W}'\boldsymbol{W}\boldsymbol{a} = \boldsymbol{W}'\boldsymbol{y} \iff \boldsymbol{W}\hat{\boldsymbol{\alpha}} = \boldsymbol{P_W}\boldsymbol{y}.$$

$$\begin{aligned} \therefore \boldsymbol{X}'\boldsymbol{X}(\boldsymbol{T}\hat{\boldsymbol{\alpha}}) &= \boldsymbol{X}'(\boldsymbol{X}\boldsymbol{T})\hat{\boldsymbol{\alpha}} \\ &= \boldsymbol{X}'\boldsymbol{W}\hat{\boldsymbol{\alpha}} \\ &= \boldsymbol{X}'\boldsymbol{P_W}\boldsymbol{y} \\ &= \boldsymbol{X}'\boldsymbol{P_X}\boldsymbol{y} \\ &= \boldsymbol{X}'\boldsymbol{y}. \end{aligned}$$

□

Example:

Suppose

$$\boldsymbol{y} = \begin{bmatrix} y_{11} \\ y_{12} \\ \vdots \\ y_{1n_1} \\ y_{21} \\ y_{22} \\ \vdots \\ y_{2n_2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{bmatrix}.$$

Consider

$$\boldsymbol{X} = \begin{bmatrix} \mathbf{1}_{n_1} & \mathbf{1}_{n_1} & \mathbf{0}_{n_1} \\ \mathbf{1}_{n_2} & \mathbf{0}_{n_2} & \mathbf{1}_{n_2} \end{bmatrix} \quad \text{and} \quad \boldsymbol{W} = \begin{bmatrix} \mathbf{1}_{n_1} & \mathbf{0}_{n_1} \\ \mathbf{1}_{n_2} & \mathbf{1}_{n_2} \end{bmatrix}.$$

Find $\boldsymbol{T} \ni \boldsymbol{W} = \boldsymbol{X}\boldsymbol{T}$.

Find $\hat{\boldsymbol{\beta}}$ a solution to $\boldsymbol{X}'\boldsymbol{X}\boldsymbol{b} = \boldsymbol{X}'\boldsymbol{y}$.

Find $\hat{\boldsymbol{\alpha}}$ a solution to $\boldsymbol{W}'\boldsymbol{W}\boldsymbol{a} = \boldsymbol{W}'\boldsymbol{y}$.

Show that $\boldsymbol{X}\hat{\boldsymbol{\beta}} = \boldsymbol{W}\hat{\boldsymbol{\alpha}}$.

$$\begin{bmatrix} \mathbf{1}_{n_1} & \mathbf{1}_{n_1} & \mathbf{0}_{n_1} \\ \mathbf{1}_{n_2} & \mathbf{0}_{n_2} & \mathbf{1}_{n_2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{1}_{n_1} & \mathbf{0}_{n_1} \\ \mathbf{1}_{n_2} & \mathbf{1}_{n_2} \end{bmatrix}$$

$$\boldsymbol{X}\boldsymbol{T} = \boldsymbol{W}.$$

$$X'X = \begin{bmatrix} n_1 + n_2 & n_1 & n_2 \\ n_1 & n_1 & 0 \\ n_2 & 0 & n_2 \end{bmatrix}.$$

One GI of $X'X$ is

$$(X'X)^- = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{n_1} & 0 \\ 0 & 0 & \frac{1}{n_2} \end{bmatrix}.$$

$$
\boldsymbol{X}'\boldsymbol{y} = \begin{bmatrix} \mathbf{1}'_{n_1} & \mathbf{1}'_{n_2} \\ \mathbf{1}'_{n_1} & \mathbf{0}'_{n_2} \\ \mathbf{0}'_{n_1} & \mathbf{1}'_{n_2} \end{bmatrix} \begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{bmatrix}
$$

$$
= \begin{bmatrix} \sum_{i=1}^{2} \sum_{j=1}^{n_i} y_{ij} \\ \sum_{j=1}^{n_1} y_{1j} \\ \sum_{j=1}^{n_2} y_{2j} \end{bmatrix} = \begin{bmatrix} y_{\cdot\cdot} \\ y_{1\cdot} \\ y_{2\cdot} \end{bmatrix}
$$

$\therefore$ a solution to $\boldsymbol{X}'\boldsymbol{X}\boldsymbol{b} = \boldsymbol{X}'\boldsymbol{y}$ is

$$(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{y} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{n_1} & 0 \\ 0 & 0 & \frac{1}{n_2} \end{bmatrix} \begin{bmatrix} \sum_{i=1}^{2} \sum_{j=1}^{n_i} y_{ij} \\ \sum_{j=1}^{n_1} y_{1j} \\ \sum_{j=1}^{n_2} y_{2j} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \bar{y}_{1\cdot} \\ \bar{y}_{2\cdot} \end{bmatrix} \equiv \hat{\boldsymbol{\beta}}.$$

$$\therefore \boldsymbol{X}\hat{\boldsymbol{\beta}} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} 0 \\ \bar{y}_{1\cdot} \\ \bar{y}_{2\cdot} \end{bmatrix}$$

$$= \begin{bmatrix} \bar{y}_{1\cdot}\mathbf{1}_{n_1} \\ \bar{y}_{2\cdot}\mathbf{1}_{n_2} \end{bmatrix} = \begin{bmatrix} \bar{y}_{1\cdot} \\ \bar{y}_{1\cdot} \\ \vdots \\ \bar{y}_{1\cdot} \\ \bar{y}_{2\cdot} \\ \bar{y}_{2\cdot} \\ \vdots \\ \bar{y}_{2\cdot} \end{bmatrix}.$$

$$\boldsymbol{W}'\boldsymbol{W} = \begin{bmatrix} n_1 + n_2 & n_2 \\ n_2 & n_2 \end{bmatrix}$$

$$(\boldsymbol{W}'\boldsymbol{W})^{-1} = \frac{1}{(n_1+n_2)n_2 - n_2^2} \begin{bmatrix} n_2 & -n_2 \\ -n_2 & n_1 + n_2 \end{bmatrix}$$

$$= \frac{1}{n_1 n_2} \begin{bmatrix} n_2 & -n_2 \\ -n_2 & n_1 + n_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{n_1} & -\frac{1}{n_1} \\ -\frac{1}{n_1} & \frac{1}{n_1} + \frac{1}{n_2} \end{bmatrix}.$$

$$\boldsymbol{W}'\boldsymbol{y} = \begin{bmatrix} \mathbf{1}' & \mathbf{1}' \\ \mathbf{0}' & \mathbf{1}' \end{bmatrix} \begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{bmatrix}$$

$$= \begin{bmatrix} y_{\cdot\cdot} \\ y_{2\cdot} \end{bmatrix}.$$

$\therefore$ the unique solution to $\boldsymbol{W}'\boldsymbol{W}\boldsymbol{a} = \boldsymbol{W}'\boldsymbol{y}$ is

$$\begin{aligned}
\hat{\boldsymbol{\alpha}} &= (\boldsymbol{W}'\boldsymbol{W})^{-1}\boldsymbol{W}'\boldsymbol{y} \\
&= \begin{bmatrix} \frac{1}{n_1} & -\frac{1}{n_1} \\ -\frac{1}{n_1} & \frac{1}{n_1} + \frac{1}{n_2} \end{bmatrix} \begin{bmatrix} y_{\cdot\cdot} \\ y_{2\cdot} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{n_1}(y_{1\cdot} + y_{2\cdot}) - \frac{1}{n_1} y_{2\cdot} \\ -\frac{1}{n_1}(y_{1\cdot} + y_{2\cdot}) + \left(\frac{1}{n_1} + \frac{1}{n_2}\right) y_{2\cdot} \end{bmatrix} = \begin{bmatrix} \bar{y}_{1\cdot} \\ \bar{y}_{2\cdot} - \bar{y}_{1\cdot} \end{bmatrix}.
\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{W}\hat{\boldsymbol{\alpha}} &= \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \bar{y}_{1\cdot} \\ \bar{y}_{2\cdot} - \bar{y}_{1\cdot} \end{bmatrix} \\
&= \begin{bmatrix} \bar{y}_{1\cdot} \\ \bar{y}_{2\cdot} \end{bmatrix} \\
&= \boldsymbol{X}\hat{\boldsymbol{\beta}}.
\end{aligned}
$$

$$\boldsymbol{T}\hat{\boldsymbol{\alpha}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{y}_{1\cdot} \\ \bar{y}_{2\cdot} - \bar{y}_{1\cdot} \end{bmatrix}$$

$$= \begin{bmatrix} \bar{y}_{1\cdot} \\ 0 \\ \bar{y}_{2\cdot} - \bar{y}_{1\cdot} \end{bmatrix}.$$

$$\therefore \boldsymbol{X}'\boldsymbol{X}\boldsymbol{T}\hat{\boldsymbol{\alpha}} = \begin{bmatrix} n_1+n_2 & n_1 & n_2 \\ n_1 & n_1 & 0 \\ n_2 & 0 & n_2 \end{bmatrix} \begin{bmatrix} \bar{y}_{1\cdot} \\ 0 \\ \bar{y}_{2\cdot} - \bar{y}_{1\cdot} \end{bmatrix}$$

$$= \begin{bmatrix} (n_1+n_2)\bar{y}_{1\cdot} + n_2(\bar{y}_{2\cdot} - \bar{y}_{1\cdot}) \\ n_1\bar{y}_{1\cdot} \\ n_2\bar{y}_{1\cdot} + n_2(\bar{y}_{2\cdot} - \bar{y}_{1\cdot}) \end{bmatrix}$$

$$= \begin{bmatrix} y_{\cdot\cdot} \\ y_{1\cdot} \\ y_{2\cdot} \end{bmatrix} = \boldsymbol{X}'\boldsymbol{y}.$$