Introduction to the General Linear Model
Consider the General Linear Model

\[ y = X\beta + \varepsilon, \]

where \ldots
$y$ is an $n \times 1$ random vector of responses that can be observed.

The values in $y$ are values of the response variable.
$X$ is an $n \times p$ matrix of known constants.

Each column of $X$ contains the values for an explanatory variable that is also known as a predictor variable or a regressor variable in the context of multiple regression.

The matrix $X$ is sometimes referred to as the design matrix.
$\beta$ is an unknown parameter vector in $\mathbb{R}^p$.

We are often interested in estimating a LC of the elements in $\beta$ ($c'\beta$) or multiple LCs ($C\beta$) for some known $c$ or $C$. 
$\varepsilon$ is an $n \times 1$ random vector that cannot be observed.

The values in $\varepsilon$ are called errors. Initially, we assume only $E(\varepsilon) = 0$. 
Note that

\[ E(\varepsilon) = 0 \Rightarrow E(y) = E(X\beta + \varepsilon) \]
\[ = X\beta + E(\varepsilon) \]
\[ = X\beta \in C(X). \]
Thus, this general linear model simply says that $y$ is a random vector with mean $E(y)$ in the column space of $X$. 
Example:

Suppose 5 hogs are fed diet 1 for three weeks. Let $y_i$ be the weight gain of the $i^{th}$ hog for $i = 1, \ldots, 5$. 
Suppose that we assume $E(y_i) = \mu \ \forall \ i = 1, \ldots, 5$. Then

$$
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4 \\
y_5
\end{bmatrix} =
\begin{bmatrix}
1 \\
1 \\
1 \\
1 \\
1
\end{bmatrix} \mu +
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4 \\
\varepsilon_5
\end{bmatrix}
$$

where $E(\varepsilon_i) = 0 \ \forall \ i = 1, \ldots, 5$. 
Example:

Suppose 10 hogs are randomly divided into two groups of 5 hogs each. Group 1 is fed diet 1 for three weeks, group 2 is fed diet 2 for three weeks.
Let $y_{ij}$ denote the weight gain of the $j^{th}$ hog in the $i^{th}$ diet group $(i = 1, 2; j = 1, \ldots, 5)$.

If we assume that $E(y_{ij}) = \mu_i$ for $i = 1, 2; j = 1, \ldots, 5$, then
\[
\begin{bmatrix}
y_{11} \\
y_{12} \\
y_{13} \\
y_{14} \\
y_{15} \\
y_{21} \\
y_{22} \\
y_{23} \\
y_{24} \\
y_{25}
\end{bmatrix}
= \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}
\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}
+ \begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{12} \\
\varepsilon_{13} \\
\varepsilon_{14} \\
\varepsilon_{15} \\
\varepsilon_{21} \\
\varepsilon_{22} \\
\varepsilon_{23} \\
\varepsilon_{24} \\
\varepsilon_{25}
\end{bmatrix}
\]

where \( E(\varepsilon_{ij}) = 0 \ \forall \ i = 1, 2; \ j = 1, \ldots, 5. \)
Alternatively, we could assume \( E(y_{ij}) = \mu + \tau_i \ \forall \ i = 1, 2; \ j = 1, \ldots, 5. \)

Then the design matrix and parameter vector become . . .
\[
\begin{bmatrix}
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 0 & 1 \\
1 & 0 & 1 \\
1 & 0 & 1 \\
1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\mu \\
\tau_1 \\
\tau_2
\end{bmatrix}
= 
\begin{bmatrix}
\mu + \tau_1 \\
\mu + \tau_1 \\
\mu + \tau_1 \\
\mu + \tau_1 \\
\mu + \tau_2 \\
\mu + \tau_2 \\
\mu + \tau_2 \\
\mu + \tau_2
\end{bmatrix}.
\]
These models are equivalent because both design matrices have the same column space:

\[
\begin{bmatrix}
a \\
a \\
a \\
a \\
a \\
b \\
b \\
b \\
b
\end{bmatrix} : a, b \in \mathbb{R}
\]
Example:

10 steer carcasses were assigned to be measured for pH at one of five times after slaughter. Data are as follows (Schwenke & Milliken (1991), Biometrics, 47, 563-573.)
<table>
<thead>
<tr>
<th>Steer</th>
<th>Hours after Slaughter</th>
<th>pH</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>7.02</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>6.93</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>6.42</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>6.51</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>6.07</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>5.99</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>5.59</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>5.80</td>
</tr>
<tr>
<td>9</td>
<td>5</td>
<td>5.51</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>5.36</td>
</tr>
</tbody>
</table>
∀ i = 1, ... , 10, let

\[ x_i = \text{measurement time (hours after slaughter) for steer } i \]
\[ y_i = \text{pH for steer } i. \]

Suppose \( y_i = \beta_0 + \beta_1 \log(x_i) + \varepsilon_i \) where \( E(\varepsilon_i) = 0 \) ∀ \( i = 1, \ldots, 10 \).

Determine \( y \), \( X \), and \( \beta \).
Chapter 1 of our text contains many more examples. Please read the entire chapter.