Idempotency and Projection Matrices
A square matrix $P$ is idempotent iff $PP = P$. 
A square matrix $P$ is a projection matrix that projects onto the vector space $S$ iff

(a) $P$ is idempotent,

(b) $Px \in S \ \forall \ x$, and

(c) $Pz = z \ \forall \ z \in S$. 
Result P.1:

Suppose $P$ is an idempotent matrix. Prove that $P$ projects onto a vector space $S$ iff $S = C(P)$. 
Result A.14:

$AA^-$ is a projection matrix that projects onto $\mathcal{C}(A)$. 
Result A.15:

$I - A^{-1}A$ is a projection matrix that projects onto $\mathcal{N}(A)$. 
Prove that $\mathcal{C}(I - A^{-}A) = \mathcal{N}(A)$. 
Result A.16:

Any symmetric and idempotent matrix $P$ is the unique symmetric projection matrix that projects onto $C(P)$. 
Proof of Result A.16:

Suppose \( Q \) is a symmetric projection matrix that projects onto \( C(P) \). Then

\[
Pz = Qz = z \quad \forall \ z \in C(P)
\]

\[
\Rightarrow PPx = QPx \quad \forall \ x
\]

\[
\Rightarrow Px = QPx \quad \forall \ x
\]

\[
\Rightarrow P = QP.
\]
Now $Q$ is a projection matrix that projects on $C(P)$, therefore, $C(P) = C(Q)$. Thus

$$Qz = Pz = z \quad \forall \ z \in C(Q)$$

$$\Rightarrow QQx = PQx \quad \forall \ x$$

$$\Rightarrow Qx = PQx \quad \forall \ x$$

$$\Rightarrow Q = PQ.$$
Now note that

\[(P - Q)'(P - Q) = P'P - P'Q - Q'P + Q'Q\]

\[= PP - PQ - QP + QQ\]

\[= P - Q - P + Q\]

\[= 0.\]

\[\therefore P - Q = 0 \Rightarrow P = Q.\]
Any symmetric, idempotent matrix $P$ is known as an orthogonal projection matrix because $(Px) \perp (x - Px)$, i.e.,

\[
(Px)'(x - Px) = x'Px - x'P'Px \\
= x'Px - x'PPx \\
= x'Px - x'Px \\
= 0.
\]
Corollary A.4:

If $\mathbf{P}$ is a symmetric projection matrix, then $\mathbf{I} - \mathbf{P}$ is a symmetric projection matrix that projects onto $\mathcal{C}(\mathbf{P})^\perp = \mathcal{N}(\mathbf{P})$. 
Suppose $A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

- Find the orthogonal projection matrix that projects onto $C(A)$.
- Find the orthogonal projection matrix that projects onto $N(A')$.
- Find the orthogonal projection of $x = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ onto $C(A)$ and onto $N(A')$. 