Idempotency and Projection Matrices

A square matrix $\boldsymbol{P}$ is idempotent iff $\boldsymbol{PP} = \boldsymbol{P}$.

A square matrix $\boldsymbol{P}$ is a <u>projection matrix</u> that projects onto the vector space $\mathcal{S}$ iff

(a) $\boldsymbol{P}$ is idempotent,

(b) $\boldsymbol{P}\boldsymbol{x} \in \mathcal{S} \ \forall \ \boldsymbol{x}$, and

(c) $\boldsymbol{P}\boldsymbol{z} = \boldsymbol{z} \ \forall \ \boldsymbol{z} \in \mathcal{S}$.

Result P.1:

Suppose $\boldsymbol{P}$ is an idempotent matrix. Prove that $\boldsymbol{P}$ projects onto a vector space $\mathcal{S}$ iff $\mathcal{S} = \mathcal{C}(\boldsymbol{P})$.

Result A.14:

$\boldsymbol{A}\boldsymbol{A}^-$ is a projection matrix that projects onto $\mathcal{C}(\boldsymbol{A})$.

Result A.15:

$\boldsymbol{I} - \boldsymbol{A}^{-}\boldsymbol{A}$ is a projection matrix that projects onto $\mathcal{N}(\boldsymbol{A})$.

Prove that $\mathcal{C}(\boldsymbol{I} - \boldsymbol{A}^{-}\boldsymbol{A}) = \mathcal{N}(\boldsymbol{A})$.

Result A.16:

Any symmetric and idempotent matrix $\boldsymbol{P}$ is the unique symmetric projection matrix that projects onto $\mathcal{C}(\boldsymbol{P})$.

Proof of Result A.16:

Suppose $\boldsymbol{Q}$ is a symmetric projection matrix that projects onto $\mathcal{C}(\boldsymbol{P})$. Then

$$\begin{aligned}
&\boldsymbol{P}\boldsymbol{z} = \boldsymbol{Q}\boldsymbol{z} = \boldsymbol{z} \ \forall \ \boldsymbol{z} \in \mathcal{C}(\boldsymbol{P}) \\
&\Rightarrow \boldsymbol{P}\boldsymbol{P}\boldsymbol{x} = \boldsymbol{Q}\boldsymbol{P}\boldsymbol{x} \ \forall \ \boldsymbol{x} \\
&\Rightarrow \boldsymbol{P}\boldsymbol{x} = \boldsymbol{Q}\boldsymbol{P}\boldsymbol{x} \ \forall \ \boldsymbol{x} \\
&\Rightarrow \boldsymbol{P} = \boldsymbol{Q}\boldsymbol{P}.
\end{aligned}$$

Now $\boldsymbol{Q}$ is a projection matrix that projects on $\mathcal{C}(\boldsymbol{P})$, therefore, $\mathcal{C}(\boldsymbol{P}) = \mathcal{C}(\boldsymbol{Q})$. Thus

$$\begin{aligned}
\boldsymbol{Qz} &= \boldsymbol{Pz} = \boldsymbol{z} \;\forall\; \boldsymbol{z} \in \mathcal{C}(\boldsymbol{Q}) \\
&\Rightarrow \boldsymbol{QQx} = \boldsymbol{PQx} \;\forall\; \boldsymbol{x} \\
&\Rightarrow \boldsymbol{Qx} = \boldsymbol{PQx} \;\forall\; \boldsymbol{x} \\
&\Rightarrow \boldsymbol{Q} = \boldsymbol{PQ}.
\end{aligned}$$

Now note that

$$
\begin{aligned}
(\boldsymbol{P}-\boldsymbol{Q})'(\boldsymbol{P}-\boldsymbol{Q}) &= \boldsymbol{P}'\boldsymbol{P}-\boldsymbol{P}'\boldsymbol{Q}-\boldsymbol{Q}'\boldsymbol{P}+\boldsymbol{Q}'\boldsymbol{Q} \\
&= \boldsymbol{P}\boldsymbol{P}-\boldsymbol{P}\boldsymbol{Q}-\boldsymbol{Q}\boldsymbol{P}+\boldsymbol{Q}\boldsymbol{Q} \\
&= \boldsymbol{P}-\boldsymbol{Q}-\boldsymbol{P}+\boldsymbol{Q} \\
&= \boldsymbol{0}.
\end{aligned}
$$

$\therefore \boldsymbol{P}-\boldsymbol{Q}=\boldsymbol{0} \Rightarrow \boldsymbol{P}=\boldsymbol{Q}.$ $\square$

Any symmetric, idempotent matrix $\boldsymbol{P}$ is known as an orthogonal projection matrix because $(\boldsymbol{P}\boldsymbol{x}) \perp (\boldsymbol{x} - \boldsymbol{P}\boldsymbol{x})$, i.e.,

$$
\begin{aligned}
(\boldsymbol{P}\boldsymbol{x})'(\boldsymbol{x} - \boldsymbol{P}\boldsymbol{x}) &= \boldsymbol{x}'\boldsymbol{P}\boldsymbol{x} - \boldsymbol{x}'\boldsymbol{P}'\boldsymbol{P}\boldsymbol{x} \\
&= \boldsymbol{x}'\boldsymbol{P}\boldsymbol{x} - \boldsymbol{x}'\boldsymbol{P}\boldsymbol{P}\boldsymbol{x} \\
&= \boldsymbol{x}'\boldsymbol{P}\boldsymbol{x} - \boldsymbol{x}'\boldsymbol{P}\boldsymbol{x} \\
&= 0.
\end{aligned}
$$

Corollary A.4:

If $\boldsymbol{P}$ is a symmetric projection matrix, then $\boldsymbol{I} - \boldsymbol{P}$ is a symmetric projection matrix that projects onto $\mathcal{C}(\boldsymbol{P})^{\perp} = \mathcal{N}(\boldsymbol{P})$.

Suppose $\boldsymbol{A} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

- Find the orthogonal projection matrix that projects onto $\mathcal{C}(\boldsymbol{A})$.

- Find the orthogonal projection matrix that projects onto $\mathcal{N}(\boldsymbol{A}')$.

- Find the orthogonal projection of $\boldsymbol{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ onto $\mathcal{C}(\boldsymbol{A})$ and onto $\mathcal{N}(\boldsymbol{A}')$.