Idempotency and Projection Matrices
A square matrix $P$ is idempotent iff $PP = P$. 
A square matrix $P$ is a projection matrix that projects onto the vector space $S \subseteq \mathbb{R}^n$ iff

(a) $P$ is idempotent,

(b) $Px \in S \forall x \in \mathbb{R}^n$, and

(c) $Pz = z \forall z \in S$. 

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Result P.1:

Suppose $P$ is an idempotent matrix. Prove that $P$ projects onto a vector space $S$ iff $S = C(P)$. 
Proof of Result P.1:

\(\iff\) Property (b) of a projection matrix implies that

\[
Px \in S \forall x \therefore \mathcal{C}(P) \subseteq S.
\]

By Property (c) of a projection matrix, \(Pz = z\ \forall z \in S\).

Thus, any \(z \in S\) also in \(\mathcal{C}(P)\). \(\therefore S \subseteq \mathcal{C}(P)\), and we have \(\mathcal{C}(P) = S\).
Need to show that any idempotent $P$ is a projection matrix that projects onto $C(P)$ as follows:

(a) $PP = P$,

(b) $Px \in C(P) \ \forall \ x$,

(c) $z \in C(P) \Rightarrow \exists \ x \ni z = Px$. Therefore, $Pz = PPx = Px = z$. $\square$
Result A.14:

$AA^{-}$ is a projection matrix that projects onto $\mathcal{C}(A)$. 
Proof of Result A.14:

(a) \((AA^-)(AA^-) = (AA^-A)A^- = AA^-\). Therefore, \(AA^-\) is idempotent.

(b) \(AA^-x = Az \quad \forall \ x\), where \(z = A^-x\). Thus \(AA^-x \in C(A) \quad \forall \ x\).

(c) \(\forall \ z \in C(A), \exists \ y \ni z = Ay\), \(\therefore AA^-z = AA^-Ay = Ay = z\). \(\square\)
Alternatively, we could have proved idempotency and then shown $C(A) = C(AA^-)$ as below:

\[ Ax = (AA^-A)x = (AA^-)Ax \Rightarrow C(A) \subseteq C(AA^-). \]

\[ AA^-x = A(A^-x) \Rightarrow C(AA^-) \subseteq C(A). \]

\[ \therefore C(A) = C(AA^-). \]
Result A.15:

$I - A^{-1}A$ is a projection matrix that projects onto $\mathcal{N}(A)$. 
Proof of Result A.15:

(a)

\[(I - A^{-}A)(I - A^{-}A)\]
\[= I - A^{-}A - A^{-}A + A^{-}AA^{-}A\]
\[= I - A^{-}A - A^{-}A + A^{-}A\]
\[= I - A^{-}A.\]
(b) Note that

\[ A(I - A^-A)x = (A - AA^-A)x \]

\[ = (A - A)x \]

\[ = 0 \quad \forall \ x. \]

\[ \therefore (I - A^-A)x \in \mathcal{N}(A) \quad \forall \ x. \]
(c) If \( z \in \mathcal{N}(A) \), then

\[
(I - A^{-}A)z = z - A^{-}Az
= z - 0
= z.
\]

\[\square\]
Prove that $C(I - A^{-1}A) = N(A)$. 
Proof:

The result follows from Result A.15 and P.1.

An alternative proof is as follows.
Proof:

Suppose $z \in \mathcal{N}(A)$. Then

$$Az = 0 \Rightarrow A^-Az = 0$$

$$\Rightarrow z - A^-Az = z$$

$$\Rightarrow (I - A^-A)z = z$$

$$\Rightarrow z \in \mathcal{C}(I - A^-A).$$

$\therefore \mathcal{N}(A) \subseteq \mathcal{C}(I - A^-A).$
Suppose $z \in C(I - A^{-}A)$. Then $\exists x \ni z = (I - A^{-}A)x$. Thus

$$Az = A(I - A^{-}A)x$$
$$= (A - AA^{-}A)x$$
$$= (A - A)x$$
$$= 0.$$ 

Thus, $z \in \mathcal{N}(A)$. It follows that $C(I - A^{-}A) \subseteq \mathcal{N}(A)$. Hence, $C(I - A^{-}A) = \mathcal{N}(A)$.
Result A.16:

Any symmetric and idempotent matrix $P$ is the unique symmetric projection matrix that projects onto $C(P)$. 
Proof of Result A.16:

Suppose $Q$ is a symmetric projection matrix that projects onto $C(P)$. Then

$$Pz = Qz = z \quad \forall \ z \in C(P)$$

$$\Rightarrow PPx = QPx \quad \forall \ x$$

$$\Rightarrow Px = QPx \quad \forall \ x$$

$$\Rightarrow P = QP.$$
Now $Q$ is a projection matrix that projects on $C(P)$, therefore, $C(P) = C(Q)$. Thus

$$Qz = Pz = z \; \forall \; z \in C(Q)$$

$$\Rightarrow QQx = PQx \; \forall \; x$$

$$\Rightarrow Qx = PQx \; \forall \; x$$

$$\Rightarrow Q = PQ.$$
Now note that

\[(P - Q)'(P - Q) = P'P - P'Q - Q'P + Q'Q\]

\[= PP - PQ - QP + QQ\]

\[= P - Q - P + Q\]

\[= 0.\]

\[\therefore P - Q = 0 \Rightarrow P = Q.\]
Any symmetric, idempotent matrix $P$ is known as an orthogonal projection matrix because $(Px) \perp (x - Px)$, i.e.,

$$(Px)'(x - Px) = x'Px - x'P'Px$$

$$= x'Px - x'PPx$$

$$= x'Px - x'Px$$

$$= 0.$$
Corollary A.4:

If $P$ is a symmetric projection matrix, then $I - P$ is a symmetric projection matrix that projects onto $\mathcal{C}(P) = \mathcal{N}(P)$. 
Proof of Corollary A.4:

First note that $\mathcal{C}(P) = \mathcal{N}(P') = \mathcal{N}(P)$ by the symmetry of $P$.

We need to show that properties (a-c) of a projection matrix hold for $I - P$ onto $\mathcal{N}(P)$. 
(a) Is $I - P$ idempotent?

$$(I - P)(I - P) = I - P - P + PP$$

$$= I - P - P + P$$

$$= I - P.$$
(b) Is \((I - P)x \in \mathcal{N}(P) \ \forall \ x?\)

\[
P(I - P)x = (P - PP)x
\]
\[
= (P - P)x
\]
\[
= 0.
\]

\[\therefore (I - P)x \in \mathcal{N}(P) \ \forall \ x.\]
(c) Does $(I - P)z = z \forall z \in \mathcal{N}(P)$?

\[ \forall z \in \mathcal{N}(P), \ (I - P)z = z - Pz \]
\[ = z - 0 \]
\[ = z. \]

Finally, we should note that $(I - P)' = I' - P' = I - P$ so that $I - P$ is symmetric as claimed in statement of the result. \[ \square \]
Suppose \( A = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \).

- Find the orthogonal projection matrix that projects onto \( \mathcal{C}(A) \).
- Find the orthogonal projection matrix that projects onto \( \mathcal{N}(A') \).
- Find the orthogonal projection of \( x = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \) onto \( \mathcal{C}(A) \) and onto \( \mathcal{N}(A') \).
Need to find a symmetric, idempotent matrix whose column space is \( C(A) \), where

\[
C(A) = \{ x \in \mathbb{R}^2 : x_1 = x_2 \}.
\]

Thus, \( P \) must have the form

\[
P = \begin{bmatrix}
a & a \\
a & a
\end{bmatrix}.
\]
Because $P$ must be idempotent,

$$
\begin{bmatrix}
 a & a \\
 a & a \\
\end{bmatrix}
\begin{bmatrix}
 a & a \\
 a & a \\
\end{bmatrix} =
\begin{bmatrix}
 2a^2 & 2a^2 \\
 2a^2 & 2a^2 \\
\end{bmatrix} =
\begin{bmatrix}
 a & a \\
 a & a \\
\end{bmatrix}.
$$

This implies $2a^2 = a \Rightarrow a = 1/2$. \therefore $P = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$.
We know

\[
I - P = \begin{bmatrix}
1/2 & -1/2 \\
-1/2 & 1/2 \\
\end{bmatrix}
\]

is the orthogonal projection matrix that projects onto

\[C(P) = C(A) = N(A').\]
\[ P \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad (I - P) \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \]