Miscellaneous Results, Solving Equations, and Generalized Inverses
Result A.7:

Suppose $S$ and $T$ are vector spaces. If $S \subseteq T$ and $\dim(S) = \dim(T)$, then $S = T$. 
Result A.8:

Suppose $A$ and $b$ satisfy

$$Ax + b = 0 \quad \forall x \in \mathbb{R}^n.$$ 

Then $A = 0$ and $b = 0$. 
Corollary A.1:

If \( B \) and \( C \) satisfy \( Bx = Cx \ \forall \ x \in \mathbb{R}^n \), then \( B = C \).
Corollary A.2:

Suppose $A_{m \times n}$ has full column rank. Then

$$AB = AC \implies B = C.$$
Lemma A.1:

\[ C'C = 0 \implies C = 0. \]

Proof of Lemma A.1:

- Let \( c_i \) denote the \( i^{th} \) column of \( C \). Then the \( i^{th} \) diagonal element of \( C'C \) is \( c_i'c_i \).

\[ \therefore C'C = 0, c_i'c_i = 0 \quad \forall \ i = 1, \ldots, n. \]

- Now \( c_i'c_i = 0 \quad \forall \ i \implies c_i = 0 \quad \forall \ i \implies C = 0. \]
Another Result on the Rank of a Product

Suppose \( \text{rank}(A_{m \times n}) = n \) and \( \text{rank}(C_{k \times l}) = k \). Then

\[
\text{rank}(B_{n \times k}) = \text{rank}(ABC).
\]

Proof: HW problem.

Corollaries:

- If \( A \) is full-column rank, \( \text{rank}(AB) = \text{rank}(B) \).

- If \( C \) is full-row rank, \( \text{rank}(BC) = \text{rank}(B) \).
Solving Equations:

Consider a system of linear equations

\[ Ax = c, \]

where \( A \) is a known matrix and \( c \) is a known vector.
• We seek a solution vector $x$ that satisfies $Ax = c$.

• If $m = n$ so that $A$ is a square and if $A$ is nonsingular, then

$$A^{-1}Ax = A^{-1}c \implies x = A^{-1}c$$

is the unique solution to $Ax = c$. 
If $A_{m \times n}$ is singular or not square, $Ax = c$ may have no solution or infinitely many solutions or a unique solution.
A system of equations $Ax = c$ is **consistent** if there exists a solution $x^*$ such that $Ax^* = c$.

A systems of equation $Ax = c$ is **inconsistent** if $Ax \neq c \ \forall \ x \in \mathbb{R}^n$. 
Result A.9:

A system of equations $Ax = c$ is consistent iff $c \in C(A)$.

- Provide an example $A, \ c \notin A x = c$ is inconsistent.

- Provide an example $A, \ c \in A x = c$ is consistent.
Generalized Inverse

- A matrix $G$ is a generalized inverse (GI) of a matrix $A$ iff

$$AGA = A.$$ 

- Every matrix has at least one GI. We will use $A^{-}$ to denote a GI of a matrix $A$. 
If $A$ is nonsingular, then $A^{-1}$ is the unique GI of $A$. 
Result A.10:

Suppose $\text{rank}(A) = r$. If $A$ can be partitioned as

$$A = \begin{bmatrix}
C & D \\
E & F
\end{bmatrix},$$

where $\text{rank}(C) = r$, then

$$G = \begin{bmatrix}
C^{-1} & 0 \\
0 & 0
\end{bmatrix}$$

is a GI of $A$. 
Proof of Result A.10:

\[
AG = \begin{bmatrix} C & D \\ E & F \end{bmatrix} \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ EC^{-1} & 0 \end{bmatrix}.
\]

\[\therefore AGA = \begin{bmatrix} I & 0 \\ EC^{-1} & 0 \end{bmatrix} \begin{bmatrix} C & D \\ E & F \end{bmatrix} = \begin{bmatrix} C & D \\ E & EC^{-1}D \end{bmatrix}.
\]

We need to show \(EC^{-1}D = F\).
First note that

\[ \text{rank}(C) = r \implies \text{rank}([C, D]) = r \]

\[ \therefore [C, D] \text{ has at least } r \text{ LI columns and at most } r \text{ LI rows.} \]

\[ (\text{rank}([C, D]) \geq r \text{ and } \text{rank}([C, D]) \leq r \implies \text{rank}([C, D]) = r.) \]
Now \( \text{rank} \left( \begin{bmatrix} C & D \\ E & F \end{bmatrix} \right) = r \implies \) each row of \([E, F]\) is a LC of the rows of \([C, D]\).

Thus \( \exists \) a matrix \( K \ni \)

\[
K[C, D] = [E, F] \\
\iff [KC, KD] = [E, F] \\
\iff KC = E, KD = F.
\]
Now $KC = E \iff K = EC^{-1}$. Together with $KD = F$, this implies $EC^{-1}D = F$. □
Permutation Matrix

A matrix $P_{n \times n}$ is a permutation matrix if the rows of $P$ are the same as the rows of $I_{n \times n}$ but not necessarily in the same order.
Example:

\[ P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \]

- If \( A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \), then \( PA = \begin{bmatrix} 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 \\ 9 & 10 & 11 & 12 \end{bmatrix} \).

- Order of rows of \( PA \) are permuted relative to order of rows of \( A \).
Example:

\[ P = \begin{bmatrix}
  0 & 1 & 0 \\
  1 & 0 & 0 \\
  0 & 0 & 1 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
  1 & 2 & 3 \\
  4 & 5 & 6 \\
\end{bmatrix}. \]

Then \[ BP = \begin{bmatrix}
  2 & 1 & 3 \\
  5 & 4 & 6 \\
\end{bmatrix}. \]

Order of columns of \[ BP \] are permuted relative to order of columns of \[ B. \]
A Permutation Matrix is Nonsingular

The rows (and columns) of a permutation matrix are the same as those of the identity matrix. Thus, a permutation matrix has full rank and is therefore nonsingular.

Furthermore, if $P = [p_1, \ldots, p_n]$ is a permutation matrix,

$$P^{-1} = P' \cdot p_i'p_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$
Result A.11:

Suppose \( \text{rank}(A) = r \). There exist permutation matrices \( P \) and \( Q \) such that

\[
PAQ = \begin{bmatrix} C & D \\ E & F \end{bmatrix},
\]

where \( \text{rank}(C) = r \). Furthermore,

\[
G = Q \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} P
\]

is a GI of \( A \).
Proof of Result A.11:

- Because $\text{rank}(A) = r$, there exists a set of $r$ rows of $A$ that are LI.

- Let $P$ be a permutation matrix, $\exists$ the first $r$ rows of $PA$ are LI.

- Let $H$ be the matrix consisting of the first $r$ rows of $PA$. Then $\text{rank}(H) = r$. 
This implies that \( \exists \) a set of \( r \) columns of \( H \) that are LI.

Let \( Q \) be a permutation matrix \( \ni \) the first \( r \) columns of \( HQ \) are LI.

Then the submatrix consisting of the first \( r \) rows and first \( r \) columns of \( PAQ \) has rank \( r \).
Thus we can partition $PAQ$ as \[
\begin{bmatrix}
C & D \\
E & F
\end{bmatrix},
\] where $\text{rank}(C) = r$.

By Result A.10, \[
\begin{bmatrix}
C^{-1} & 0 \\
0 & 0
\end{bmatrix}
\] is a GI for $PAQ = \begin{bmatrix} C & D \\ E & F \end{bmatrix}$.
\[ PAQ = \begin{bmatrix} C & D \\ E & F \end{bmatrix} \iff A = P^{-1} \begin{bmatrix} C & D \\ E & F \end{bmatrix} Q^{-1} \]

\[
\therefore AQ \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} PA = \\
= P^{-1} \begin{bmatrix} C & D \\ E & F \end{bmatrix} Q^{-1}Q \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} PP^{-1} \begin{bmatrix} C & D \\ E & F \end{bmatrix} Q^{-1} \\
= P^{-1} \begin{bmatrix} C & D \\ E & F \end{bmatrix} PP^{-1} \begin{bmatrix} C & D \\ E & F \end{bmatrix} Q^{-1} \\
= P^{-1} \begin{bmatrix} C & D \\ E & F \end{bmatrix} Q^{-1} = A. \]
Use Result A.11 to find a GI for

\[
A = \begin{bmatrix}
1 & 1 & 3 & 4 \\
2 & 2 & 6 & 8 \\
3 & 3 & 0 & 12
\end{bmatrix}.
\]
Algorithm for finding a GI of $A$:

1. Find an $r \times r$ nonsingular submatrix of $A$, where $r = \text{rank}(A)$. Call this matrix $W$.

2. Compute $(W^{-1})'$.

3. Replace each element of $W$ in $A$ with the corresponding elements of $(W^{-1})'$.

4. Replace all other elements in $A$ with zeros.

5. Transpose resulting matrix to get $A^\top$. 
Result A.12:

Let $Ax = c$ be a consistent system of equations, and let $G$ be any GI of $A$. Then $Gc$ is a solution to $Ax = c$, i.e., $AGc = c$. 
Result A.13:

Let $Ax = c$ be a consistent system of equations, and let $G$ be any GI of $A$. Then $\tilde{x}$ is a solution to $Ax = c$ iff $\exists \ z \ \exists \ \tilde{x} = Gc + (I - GA)z$. 
Prove that a consistent system of equations

\[ Ax = c \]

has a unique solution if and only if \( A \) is full-column rank and infinitely many solutions if and only if \( A \) is less than full-column rank.