Orthogonal Complements

- The orthogonal complement of a vector space $\mathcal{S} \subseteq \mathbb{R}^n$, denoted $\mathcal{S}^\perp$, is

$$\mathcal{S}^\perp = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}'\boldsymbol{y} = 0 \quad \forall\, \boldsymbol{y} \in \mathcal{S}\}.$$

- Is $\mathcal{S}^\perp$ is a vector space?

Suppose a vector space $\mathcal{S}$ is a subset of $\mathbb{R}^n$. Show the following:

(a) $\mathcal{S} \cap \mathcal{S}^{\perp} = \{\mathbf{0}\}$

(b) $dim(\mathcal{S}) + dim(\mathcal{S}^{\perp}) = n$.

Result A.4:

Suppose a vector space $\mathcal{S} \subseteq \mathbb{R}^n$. Then any $\boldsymbol{x} \in \mathbb{R}^n$ can be written as

$$\boldsymbol{x} = \boldsymbol{s} + \boldsymbol{t},$$

where $\boldsymbol{s} \in \mathcal{S}, \boldsymbol{t} \in \mathcal{S}^{\perp}$. Furthermore, the decomposition is unique.

Proof of Result A.4:

- Let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r$ be a basis for $\mathcal{S}$. Let $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k$ be a basis for $\mathcal{S}^\perp$. We know that $r + k = n$.

- We now show that $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_k$ is a set of LI vectors and is thus a basis for $\mathbb{R}^n$ by V4.

- Suppose

$$c_1\boldsymbol{a}_1 + \cdots + c_r\boldsymbol{a}_r + c_{r+1}\boldsymbol{b}_1 + \cdots + c_n\boldsymbol{b}_k = \boldsymbol{0}.$$

Then

$$c_1\boldsymbol{a}_1 + \cdots + c_r\boldsymbol{a}_r = -(c_{r+1}\boldsymbol{b}_1 + \cdots + c_n\boldsymbol{b}_k).$$

- Moreover,

$$\boldsymbol{c}_1\boldsymbol{a}_1 + \cdots + c_r\boldsymbol{a}_r \in \mathcal{S} \text{ and } -(c_{r+1}\boldsymbol{b}_1 + \cdots + c_n\boldsymbol{b}_k) \in \mathcal{S}^{\perp}.$$

- Because these two vectors are equal each other, they are in $\mathcal{S}$ and $\mathcal{S}^{\perp}$ and must be $\mathbf{0}$ $\because$ $\mathcal{S} \cap \mathcal{S}^{\perp} = \{\mathbf{0}\}$.

- By LI of $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r$, we have $c_1 = \cdots = c_r = 0$.

- By LI of $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k$, we have $c_{r+1} = \cdots = c_n = 0$.

- Thus, $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_k$ are LI and a basis for $\mathbb{R}^n$.

- This implies that any $\boldsymbol{x} \in \mathbb{R}^n$ may be written as

$$\boldsymbol{x} = d_1\boldsymbol{a}_1 + \cdots + d_r\boldsymbol{a}_r + d_{r+1}\boldsymbol{b}_1 + \cdots + d_n\boldsymbol{b}_k$$

  for some $d_1, \ldots, d_n \in \mathbb{R}$.

- If we take

$$\boldsymbol{s} = d_1\boldsymbol{a}_1 + \cdots + d_r\boldsymbol{a}_r \text{ and } \boldsymbol{t} = d_{r+1}\boldsymbol{b}_1 + \cdots + d_n\boldsymbol{b}_k,$$

  we have $\boldsymbol{s} \in \mathcal{S}, \boldsymbol{t} \in \mathcal{S}^{\perp}$ and $\boldsymbol{x} = \boldsymbol{s} + \boldsymbol{t}$.

- To prove the uniqueness, suppose

$$\boldsymbol{x} = \boldsymbol{s}_1 + \boldsymbol{t}_1 = \boldsymbol{s}_2 + \boldsymbol{t}_2,$$

  where $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathcal{S}$ and $\boldsymbol{t}_1, \boldsymbol{t}_2 \in \mathcal{S}^\perp$.

- Now show that $\boldsymbol{s}_1 = \boldsymbol{s}_2$ and $\boldsymbol{t}_1 = \boldsymbol{t}_2$ to complete the proof.

- Suppose $\boldsymbol{A} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Find and sketch $\mathcal{C}(\boldsymbol{A})$ and $\mathcal{N}(\boldsymbol{A}')$.

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Result A.5:

If $\boldsymbol{A}$ is an $m \times n$ matrix, then $\mathcal{C}(\boldsymbol{A})^{\perp} = \mathcal{N}(\boldsymbol{A}')$.

Proof of Result A.5:

We essentially made this argument in our proof that

$$dim(\mathcal{S}) + dim(\mathcal{S}^{\perp}) = n$$

if $\mathcal{S} \in \mathbb{R}^n$ is a vector space. $\square$

Result A.6:

Suppose $\mathcal{S}_1, \mathcal{S}_2$ are vector spaces in $\mathbb{R}^n$ such that $\mathcal{S}_1 \subseteq \mathcal{S}_2$. Then $\mathcal{S}_2^\perp \subseteq \mathcal{S}_1^\perp$.

Suppose $\mathcal{S}$ is a vector space in $\mathbb{R}^n$. Prove that $\mathcal{S} = (\mathcal{S}^\perp)^\perp$.