Orthogonal Complements

- The orthogonal complement of a vector space $\mathcal{S} \subseteq \mathbb{R}^n$, denoted $\mathcal{S}^\perp$, is

$$\mathcal{S}^\perp = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}'\boldsymbol{y} = 0 \quad \forall\, \boldsymbol{y} \in \mathcal{S}\}.$$

- Is $\mathcal{S}^\perp$ is a vector space?

- $\forall\, c_1, c_2 \in \mathbb{R}, \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{S}^\perp$,

$$(c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2)'\boldsymbol{y} = c_1 x_1'\boldsymbol{y} + c_2\boldsymbol{x}_2'\boldsymbol{y} = 0$$

  whenever $\boldsymbol{y} \in \mathcal{S}$.

- Thus

$$\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{S}^\perp,\ c_1, c_2 \in \mathbb{R} \Longrightarrow c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 \in \mathcal{S}^\perp.$$

- It follows that $\mathcal{S}^\perp$ is a vector space.

Suppose a vector space $\mathcal{S}$ is a subset of $\mathbb{R}^n$. Show the following:

(a) $\mathcal{S} \cap \mathcal{S}^{\perp} = \{\mathbf{0}\}$

(b) $dim(\mathcal{S}) + dim(\mathcal{S}^{\perp}) = n$.

Proof of (a)

- Suppose $\boldsymbol{x} \in \mathcal{S} \cap \mathcal{S}^{\perp}$.
- Then $\boldsymbol{x}'\boldsymbol{x} = 0$, which implies that

$$\sum_{i=1}^{n} x_i^2 = 0 \Longrightarrow x_i = 0 \quad \forall\, i = 1, \ldots, n$$

$$\Longrightarrow \boldsymbol{x} = \boldsymbol{0}.$$

$\square$

Proof of (b)

- Let $r = dim(\mathcal{S})$ and let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r$ be a basis for $\mathcal{S}$.

- If we define

$$\boldsymbol{A} = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r],$$

  then $\mathcal{C}(\boldsymbol{A}) = \mathcal{S}$.

- Now note that $\mathcal{S}^{\perp} = \mathcal{N}(\boldsymbol{A}')$ as follows:

-
$$\begin{aligned} \boldsymbol{x} \in \mathcal{S}^{\perp} &\implies \boldsymbol{x}'\boldsymbol{a}_i = 0 \quad \forall\, i = 1, \ldots, r \\ &\implies \boldsymbol{x}'[\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r] = \boldsymbol{0}' \\ &\implies \boldsymbol{x}'\boldsymbol{A} = \boldsymbol{0}' \implies \boldsymbol{A}'\boldsymbol{x} = \boldsymbol{0} \\ &\implies \boldsymbol{x} \in \mathcal{N}(\boldsymbol{A}') \therefore \mathcal{S}^{\perp} \subseteq \mathcal{N}(\boldsymbol{A}'). \end{aligned}$$

- Conversely,

$$\begin{aligned} \boldsymbol{x} \in \mathcal{N}(\boldsymbol{A}') &\implies \boldsymbol{A}'\boldsymbol{x} = \boldsymbol{0} \\ &\implies \boldsymbol{x}'\boldsymbol{A} = \boldsymbol{0}' \\ &\implies \boldsymbol{x}'\boldsymbol{A}\boldsymbol{y} = 0 \;\forall\, \boldsymbol{y} \in \mathbb{R}^r \\ &\implies \boldsymbol{x}'\boldsymbol{z} = 0 \;\forall\, \boldsymbol{z} \in \mathcal{C}(\boldsymbol{A}) = \mathcal{S} \\ &\implies \boldsymbol{x} \in \mathcal{S}^{\perp}. \end{aligned}$$

$\therefore \mathcal{N}(\boldsymbol{A}') \subseteq \mathcal{S}^{\perp}$ and we have $\mathcal{N}(\boldsymbol{A}') = \mathcal{S}^{\perp}$.

- From Theorem A.1,

$$\begin{aligned} dim(\mathcal{N}(\boldsymbol{A}')) &= n - rank(\boldsymbol{A}') \\ &= n - rank(\boldsymbol{A}) \\ &= n - r. \end{aligned}$$

- Therefore,

$$\begin{aligned} dim(\mathcal{S}) + dim(\mathcal{S}^{\perp}) &= dim(\mathcal{C}(\boldsymbol{A})) + dim(\mathcal{N}(\boldsymbol{A}')) \\ &= r + n - r \\ &= n. \end{aligned}$$

□

Result A.4:

Suppose a vector space $\mathcal{S} \subseteq \mathbb{R}^n$. Then any $\boldsymbol{x} \in \mathbb{R}^n$ can be written as

$$\boldsymbol{x} = \boldsymbol{s} + \boldsymbol{t},$$

where $\boldsymbol{s} \in \mathcal{S}, \boldsymbol{t} \in \mathcal{S}^{\perp}$. Furthermore, the decomposition is unique.

Proof of Result A.4:

- Let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r$ be a basis for $\mathcal{S}$. Let $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k$ be a basis for $\mathcal{S}^\perp$. We know that $r + k = n$.

- We now show that $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_k$ is a set of LI vectors and is thus a basis for $\mathbb{R}^n$ by V4.

- Suppose

$$c_1\boldsymbol{a}_1 + \cdots + c_r\boldsymbol{a}_r + c_{r+1}\boldsymbol{b}_1 + \cdots + c_n\boldsymbol{b}_k = \boldsymbol{0}.$$

Then

$$c_1\boldsymbol{a}_1 + \cdots + c_r\boldsymbol{a}_r = -(c_{r+1}\boldsymbol{b}_1 + \cdots + \ c_n\boldsymbol{b}_k).$$

- Moreover,

$$\boldsymbol{c}_1\boldsymbol{a}_1 + \cdots + c_r\boldsymbol{a}_r \in \mathcal{S} \text{ and } -(c_{r+1}\boldsymbol{b}_1 + \cdots + \ c_n\boldsymbol{b}_k) \in \mathcal{S}^\perp.$$

- Because these two vectors are equal each other, they are in $\mathcal{S}$ and $\mathcal{S}^\perp$ and must be $\mathbf{0} \because \mathcal{S} \cap \mathcal{S}^\perp = \{\mathbf{0}\}$.

- By LI of $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r$, we have $c_1 = \cdots = c_r = 0$.

- By LI of $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k$, we have $c_{r+1} = \cdots = c_n = 0$.

- Thus, $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_k$ are LI and a basis for $\mathbb{R}^n$.

- This implies that any $\boldsymbol{x} \in \mathbb{R}^n$ may be written as

$$\boldsymbol{x} = d_1\boldsymbol{a}_1 + \cdots + d_r\boldsymbol{a}_r + d_{r+1}\boldsymbol{b}_1 + \cdots + d_n\boldsymbol{b}_k$$

  for some $d_1, \ldots, d_n \in \mathbb{R}$.

- If we take

$$\boldsymbol{s} = d_1\boldsymbol{a}_1 + \cdots + d_r\boldsymbol{a}_r \text{ and } \boldsymbol{t} = d_{r+1}\boldsymbol{b}_1 + \cdots + d_n\boldsymbol{b}_k,$$

  we have $\boldsymbol{s} \in \mathcal{S}, \boldsymbol{t} \in \mathcal{S}^\perp$ and $\boldsymbol{x} = \boldsymbol{s} + \boldsymbol{t}$.

- To prove the uniqueness, suppose

$$\boldsymbol{x} = \boldsymbol{s}_1 + \boldsymbol{t}_1 = \boldsymbol{s}_2 + \boldsymbol{t}_2,$$

  where $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathcal{S}$ and $\boldsymbol{t}_1, \boldsymbol{t}_2 \in \mathcal{S}^\perp$.

- Now show that $\boldsymbol{s}_1 = \boldsymbol{s}_2$ and $\boldsymbol{t}_1 = \boldsymbol{t}_2$ to complete the proof.

- $\boldsymbol{s}_1 + \boldsymbol{t}_1 = \boldsymbol{s}_2 + \boldsymbol{t}_2 \Longrightarrow \boldsymbol{s}_1 - \boldsymbol{s}_2 = \boldsymbol{t}_2 - \boldsymbol{t}_1$.
- Because $\mathcal{S}$ and $\mathcal{S}^\perp$ are vectors spaces, $\boldsymbol{s}_1 - \boldsymbol{s}_2 \in \mathcal{S}, \quad \boldsymbol{t}_2 - \boldsymbol{t}_1 \in \mathcal{S}^\perp$.
- The equality of these vectors implies that

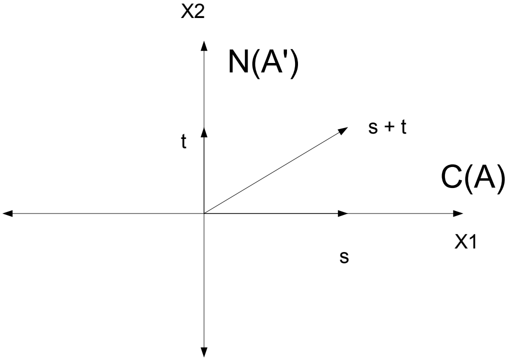
$$\boldsymbol{s}_1 - \boldsymbol{s}_2 = \boldsymbol{t}_2 - \boldsymbol{t}_1 \in \mathcal{S} \cap \mathcal{S}^\perp \Longrightarrow \boldsymbol{s}_1 - \boldsymbol{s}_2 = \boldsymbol{t}_2 - \boldsymbol{t}_1 = \boldsymbol{0}.$$

- Therefore, $\boldsymbol{s}_1 = \boldsymbol{s}_2, \quad \boldsymbol{t}_2 = \boldsymbol{t}_1$. □

- Suppose $\boldsymbol{A} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Find and sketch $\mathcal{C}(\boldsymbol{A})$ and $\mathcal{N}(\boldsymbol{A}')$.
- Suppose $\boldsymbol{A} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find and sketch $\mathcal{C}(\boldsymbol{A})$ and $\mathcal{N}(\boldsymbol{A}')$.

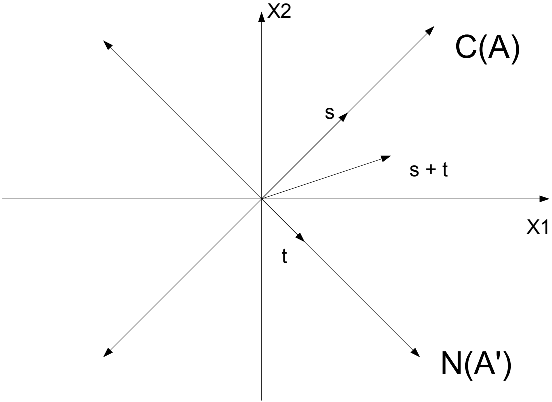
If $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $\mathcal{C}(A) = \{x \in \mathbb{R}^2 : x_2 = 0\}$.

$$\begin{aligned} \mathcal{N}(A') &= \{x \in \mathbb{R}^2 : [1, 0]x = 0\} \\ &= \{x \in \mathbb{R}^2 : x_1 = 0\}. \end{aligned}$$



If $\boldsymbol{A} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $\mathcal{C}(\boldsymbol{A}) = \{\boldsymbol{x} \in \mathbb{R}^2 : x_1 = x_2\}$.

$$\begin{aligned} \mathcal{N}(\boldsymbol{A}') &= \{\boldsymbol{x} \in \mathbb{R}^2 : [1, 1]\boldsymbol{x} = 0\} \\ &= \{\boldsymbol{x} \in \mathbb{R}^2 : x_1 + x_2 = 0\}. \end{aligned}$$



Result A.5:

If $\boldsymbol{A}$ is an $m \times n$ matrix, then $\mathcal{C}(\boldsymbol{A})^{\perp} = \mathcal{N}(\boldsymbol{A}')$.

Proof of Result A.5:

We essentially made this argument in our proof that

$$dim(\mathcal{S}) + dim(\mathcal{S}^{\perp}) = n$$

if $\mathcal{S} \in \mathbb{R}^n$ is a vector space. $\square$

Result A.6:

Suppose $\mathcal{S}_1, \mathcal{S}_2$ are vector spaces in $\mathbb{R}^n$ such that $\mathcal{S}_1 \subseteq \mathcal{S}_2$. Then $\mathcal{S}_2^\perp \subseteq \mathcal{S}_1^\perp$.

Proof of Result A.6:

- Suppose $\boldsymbol{x} \in \mathcal{S}_2^\perp$.
- Then

$$\begin{aligned} & \boldsymbol{x}'\boldsymbol{y} = 0 \quad \forall\, \boldsymbol{y} \in \mathcal{S}_2 \\ \Longrightarrow\ & \boldsymbol{x}'\boldsymbol{y} = 0 \quad \forall\, \boldsymbol{y} \in \mathcal{S}_1 \\ \Longrightarrow\ & \boldsymbol{x} \in \mathcal{S}_1^\perp. \end{aligned}$$

- We have $\mathcal{S}_2^\perp \subseteq \mathcal{S}_1^\perp$. $\square$

Suppose $\mathcal{S}$ is a vector space in $\mathbb{R}^n$. Prove that $\mathcal{S} = (\mathcal{S}^\perp)^\perp$.

Proof

- $\boldsymbol{x} \in \mathcal{S} \Longrightarrow \boldsymbol{x}'\boldsymbol{y} = 0 \; \forall \; \boldsymbol{y} \in \mathcal{S}^{\perp} \Longrightarrow \boldsymbol{x} \in (\mathcal{S}^{\perp})^{\perp}$. $\therefore \mathcal{S} \subseteq (\mathcal{S}^{\perp})^{\perp}$.
- Now suppose $\boldsymbol{x} \in (\mathcal{S}^{\perp})^{\perp}$. Then $\boldsymbol{x}'\boldsymbol{y} = 0 \; \forall \; \boldsymbol{y} \in \mathcal{S}^{\perp}$.
- By Result A.4, $\boldsymbol{x} = \boldsymbol{s} + \boldsymbol{t}$ for some $\boldsymbol{s} \in \mathcal{S}$ and some $\boldsymbol{t} \in \mathcal{S}^{\perp}$.
- By the previous two points, $0 = \boldsymbol{x}'\boldsymbol{t} = (\boldsymbol{s} + \boldsymbol{t})'\boldsymbol{t} = \boldsymbol{t}'\boldsymbol{t}$. $\therefore \; \boldsymbol{t} = \boldsymbol{0}$ and $\boldsymbol{x} = \boldsymbol{s} \in \mathcal{S}$ so that $(\mathcal{S}^{\perp})^{\perp} \subseteq \mathcal{S}$. $\square$