Preliminary Linear Algebra 1

Notation

$\forall$ for all

$\exists$ there exists

$\ni$ such that

$\therefore$ therefore

$\because$ because

$\Box$ end of proof (QED)

Notation

$A \implies B$ A implies B

$A \iff B$ A if and only if (iff) B

$a \in B$ a is an element of the set B

$A \subset B$ A is a proper subset of B

$A \subseteq B$ A is a subset of B

$\mathbb{R}^n$ Euclidean n-space

Matrix Notation

- $\underset{m\times n}{\boldsymbol{A}} = \underset{m\times n}{[a_{ij}]} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ is a matrix with m rows and n columns.
- The entry in the i^{th} row and j^{th} column of $\boldsymbol{A}$ is a_{ij}.

Square Matrices and Vectors

- Matrix $\underset{m\times n}{A}$ is said to be square if $m = n$.
- A matrix with one column is called a vector.
- A matrix with one row is called a row vector.

Notation

In these STAT611 notes, bold uppercase letters are typically used to denote matrices, and bold lowercase letters are typically used to denote vectors.

Examples

- $\mathbf{0}$ or $\mathbf{0}_n$ is a vector of zeros.
- $\mathbf{1}$ or $\mathbf{1}_n$ is a vector of ones.
- $\boldsymbol{I}$ or $\boldsymbol{I}_n$ or $\boldsymbol{I}_{n\times n}$ is the identity matrix. For example,

$$\boldsymbol{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = diag(1, 1, 1).$$

Special Types of Square Matrices

- A square matrix $\underset{n\times n}{A}$ is upper triangular if $a_{ij} = 0, \forall\ i > j$.
- A square matrix $\underset{n\times n}{A}$ is lower triangular if $a_{ij} = 0, \forall\ i < j$.
- A square matrix $\underset{n\times n}{A}$ is diagonal if $a_{ij} = 0, \forall\ i \neq j$.
- Write one example for each of these types of matrices.

Matrix Transpose

If $\underset{m\times n}{\boldsymbol{A}} = [a_{ij}]$, the transpose of $\boldsymbol{A}$, denoted $\boldsymbol{A}'$, is the matrix $\underset{n\times m}{\boldsymbol{B}} = [b_{ij}]$, where $b_{ij} = a_{ji}, \forall\ i = 1, \ldots, m; \quad j = 1, \ldots, n$.

That is, $\boldsymbol{B} = \boldsymbol{A}'$ is the matrix whose columns are the rows of $\boldsymbol{A}$ and whose rows are the columns of $\boldsymbol{A}$.

A Symmetric Matrix

A square matrix $\underset{n \times n}{\boldsymbol{A}}$ is symmetric if $\boldsymbol{A} = \boldsymbol{A}'$.

Examples

- Find the transpose of
$$\begin{bmatrix} 4 & -2 \\ 3 & 7 \end{bmatrix}.$$
- Provide an example of a symmetric matrix.

Matrix Multiplication

- Suppose

$$\underset{m\times n}{\boldsymbol{A}} = [a_{ij}] = \begin{bmatrix} \boldsymbol{a}_1' \\ \vdots \\ \boldsymbol{a}_m' \end{bmatrix} = [\boldsymbol{a}^{(1)}, \ldots, \boldsymbol{a}^{(n)}]$$

  and

$$\underset{n\times k}{\boldsymbol{B}} = [b_{ij}] = \begin{bmatrix} \boldsymbol{b}_1' \\ \vdots \\ \boldsymbol{b}_n' \end{bmatrix} = [\boldsymbol{b}^{(1)}, \ldots, \boldsymbol{b}^{(k)}].$$

- Then

$$\begin{aligned} \underset{m\times n}{\boldsymbol{A}} \; \underset{n\times k}{\boldsymbol{B}} &= \underset{m\times k}{\boldsymbol{C}} = [c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}] = [c_{ij} = \boldsymbol{a}_i' \boldsymbol{b}^{(j)}] \\ &= [\boldsymbol{A}\boldsymbol{b}^{(1)}, \ldots, \boldsymbol{A}\boldsymbol{b}^{(k)}] = \begin{bmatrix} \boldsymbol{a}_1'\boldsymbol{B} \\ \vdots \\ \boldsymbol{a}_m'\boldsymbol{B} \end{bmatrix} = \sum_{l=1}^{n} \boldsymbol{a}^{(l)} \boldsymbol{b}_l'. \end{aligned}$$

Matrix Multiplication

- Suppose
$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
and
$$\boldsymbol{B} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}.$$
- Work out $\boldsymbol{AB}$ using $\boldsymbol{AB} = \sum_{l=1}^{n} \boldsymbol{a}^{(l)} \boldsymbol{b}_l'$.

Transpose of a Matrix Product

- $$(\boldsymbol{AB})' = \boldsymbol{B}'\boldsymbol{A}'$$

- The transpose of a product is the product of the transposes in reverse order.

Scalar Multiplication of a Matrix

If $c \in \mathbb{R}$, then c times the matrix $\boldsymbol{A}$ is the matrix whose i, j^{th} element is c times the i, j^{th} element of $\boldsymbol{A}$; i.e.,

$$\underset{m \times n}{c\boldsymbol{A}} = c\underset{m \times n}{[a_{ij}]} = \underset{m \times n}{[ca_{ij}]} = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}.$$

Linear Combination

If $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in \mathbb{R}^m$ and $c_1, \ldots, c_n \in \mathbb{R}$, then

$$\sum_{i=1}^{n} c_i \boldsymbol{x}_i = c_1 \boldsymbol{x}_1 + \cdots + c_n \boldsymbol{x}_n$$

is a linear combination (LC) of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$.

The coefficients of the LC are $c_1, \ldots, c_n$.

Linear Independence and Linear Dependence

- A set of vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ is linearly independent (LI) if

$$\sum_{i=1}^{n} c_i \boldsymbol{x}_i = \boldsymbol{0} \Longleftrightarrow c_1 = \cdots = c_n = 0.$$

- A set of vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ is linearly dependent (LD) if

$$\exists\, c_1, \ldots, c_n \text{ not all } 0 \ni \sum_{i=1}^{n} c_i \boldsymbol{x}_i = \boldsymbol{0}.$$

- Prove or disprove: any set of vectors that contains the vector $\mathbf{0}$ is LD.
- Prove or disprove: The following set of vectors is LI.

$$\begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ -6 \\ 7 \end{bmatrix}.$$

Fact V1:

The nonzero vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ are LD $\Longleftrightarrow$ $\boldsymbol{x}_j$ is a LC of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{j-1}$ for some $j \in \{2, \ldots, n\}$.

Orthogonality

- The two vectors $\boldsymbol{x}, \boldsymbol{y}$ are orthogonal to each other if their inner product is zero, i.e.,

$$\boldsymbol{x}'\boldsymbol{y} = \boldsymbol{y}'\boldsymbol{x} = \sum_{i=1}^{n} x_i y_i = 0.$$

- The length of a vector, also known as its Euclidean norm, is

$$\|\boldsymbol{x}\| := \sqrt{\boldsymbol{x}'\boldsymbol{x}} = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

- The vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ are mutually orthogonal if

$$\boldsymbol{x}_i'\boldsymbol{x}_j = 0, \quad \forall\ i \neq j.$$

- The vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ are mutually orthonormal if

$$\boldsymbol{x}_i'\boldsymbol{x}_j = 0 \quad \forall\ i \neq j, \text{ and } \|\boldsymbol{x}_i\| = 1 \quad \forall\ i = 1, \ldots, n.$$

- Write down a set of mutually orthogonal but not mutually orthonormal vectors.
- Write down a set of mutually orthonormal vectors.

Orthogonal Matrix

- A square matrix with mutually orthonormal columns is called an orthogonal matrix.

- Show that if $\boldsymbol{Q}$ is orthogonal, then $\boldsymbol{Q}'\boldsymbol{Q} = \boldsymbol{I}$.
- Show that if $\boldsymbol{Q}$ is orthogonal and $\boldsymbol{x}$ is any vector of appropriate dimension, then $\|\boldsymbol{Q}\boldsymbol{x}\| = \|\boldsymbol{x}\|$.

An orthogonal matrix $\boldsymbol{Q}$ is sometimes called a rotation matrix because if a vector $\underset{n\times 1}{\boldsymbol{x}}$ is premultiplied by $\underset{n\times n}{\boldsymbol{Q}}$, the result ($\boldsymbol{Qx}$) is the vector $\boldsymbol{x}$ rotated to a new position in $\mathbb{R}^n$.

Vector Space

A vector space $\mathcal{S}$ is a set of vectors that is closed under addition (i.e., if $\boldsymbol{x}_1 \in \mathcal{S}, \boldsymbol{x}_2 \in \mathcal{S}$, then $\boldsymbol{x}_1 + \boldsymbol{x}_2 \in \mathcal{S}$) and closed under scalar multiplication (i.e., if $c \in \mathbb{R}, \boldsymbol{x} \in \mathcal{S}$, then $c\boldsymbol{x} \in \mathcal{S}$).

In other words,

$$c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 \in \mathcal{S} \quad \forall\, c_1, c_2 \in \mathbb{R};\ \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{S}.$$

- Is $\{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| = 1\}$ a vector space?

- Is $\{\boldsymbol{x} \in \mathbb{R}^n : \mathbf{1}'\boldsymbol{x} = 0\}$ a vector space?

- Is $\{\underset{n \times m}{\boldsymbol{A}}\boldsymbol{x} : \boldsymbol{x} \in \mathbb{R}^m\}$ a vector space?

Generators of a Vector Space

A vector space $\mathcal{S}$ is said to be generated by a set of vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ if

$$\boldsymbol{x} \in \mathcal{S} \Longrightarrow \boldsymbol{x} = \sum_{i=1}^{n} c_i \boldsymbol{x}_i \text{ for some } c_1, \ldots, c_n \in \mathbb{R}.$$

Span of a Set of Vectors

- The span of vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ is the set of all LC of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$, i.e.,

$$span\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\} = \left\{ \sum_{i=1}^{n} c_i \boldsymbol{x}_i : c_1, \ldots, c_n \in \mathbb{R} \right\}.$$

- $span\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$ is the vector space generated by $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$.

Find a set of vectors that generates the space

$$\{\boldsymbol{x} \in \mathbb{R}^3 : \mathbf{1}'\boldsymbol{x} = 0\};$$

i.e., find a set of vectors whose span is

$$\mathcal{S} = \{\boldsymbol{x} \in \mathbb{R}^3 : \mathbf{1}'\boldsymbol{x} = 0\}.$$

Basis of a Vector Space

If a vector space $\mathcal{S}$ is generated by LI vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$, then $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ form a basis for $\mathcal{S}$.

Fact V2:

Suppose $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ form a basis for a vector space $\mathcal{S}$. If $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k$ are LI vectors in $\mathcal{S}$, then $k \leq n$.

Proof of Fact V2:

- Because $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ form a basis for $\mathcal{S}$ and $\boldsymbol{b}_1 \in \mathcal{S}$, $\boldsymbol{b}_1 = \sum_{i=1}^{n} c_i \boldsymbol{a}_i$ for some $c_1, \ldots, c_n \in \mathbb{R}$. Thus, $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n, \boldsymbol{b}_1$ are LD by Fact V1.
- Again, using V1, we have $\boldsymbol{a}_j$ a LC of $\boldsymbol{b}_1, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{j-1}$ for some $j \in \{1, 2, \ldots, n\}$.

- Thus, $\boldsymbol{b}_1, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{j-1}, \boldsymbol{a}_{j+1}, \ldots, \boldsymbol{a}_n$ generate $\mathcal{S}$. It follows that $\boldsymbol{b}_1, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{j-1}, \boldsymbol{a}_{j+1}, \ldots, \boldsymbol{a}_n, \boldsymbol{b}_2$ is a LD set of vectors by V1.

- Again by V1, one of the vectors $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{j-1}, \boldsymbol{a}_{j+1}, \ldots, \boldsymbol{a}_n$ is a LC of the preceding vectors. It is not $\boldsymbol{b}_2 \because \boldsymbol{b}_1, \ldots, \boldsymbol{b}_k$ are LI.

- Thus $\boldsymbol{b}_1, \boldsymbol{b}_2$ and $n-2$ of $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ generate $\mathcal{S}$.

- If $k > n$, we can continue adding $\boldsymbol{b}$ vectors and deleting $\boldsymbol{a}$ vectors to get $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$ generates $\mathcal{S}$. However, then V1 would imply $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{n+1}$ are LD. This contradicts LI of $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k \therefore k \leq n$. $\square$

Fact V3:

If $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\}$ and $\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k\}$ each provide a basis for a vector space $\mathcal{S}$, then $n = k$.

Proof: By V2, we have $k \leq n$ and $n \leq k$. $\therefore k = n$. $\square$

Dimension of a Vector Space

A basis for a vector space is not unique, but the number of vectors in the basis, known as <u>dimension</u> of the vector space, is unique.

Find $dim(\mathcal{S})$, the dimension of vector space $\mathcal{S}$, for

$$\mathcal{S} = \{\boldsymbol{x} \in \mathbb{R}^3 : \mathbf{1}'\boldsymbol{x} = 0\}.$$

Consider the set $\{\underset{n\times 1}{\mathbf{0}}\}$. Is this a vector space? If so, what is its dimension?

Fact V4:

Suppose $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ are LI vectors in a vector space $\mathcal{S}$ with dimension n. Then $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ form a basis for $\mathcal{S}$.

Fact V5:

If $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k$ are LI vectors in an n-dimensional vector space $\mathcal{S}$, then there exists a basis for $\mathcal{S}$ that contains $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k$.

Fact V6:

If $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k$ are LI and orthonormal vectors in $\mathbb{R}^n$, then there exist $\boldsymbol{a}_{k+1}, \ldots, \boldsymbol{a}_n$ such that $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ are LI and orthornormal.

Proof: HW problem. □

Rank of a Matrix

It can be shown that

- the (maximum) number of LI rows of a matrix $\underset{m\times n}{\boldsymbol{A}}$ is the same as the (maximum) number of LI columns of $\underset{m\times n}{\boldsymbol{A}}$.
- This number of LI rows or columns is known as the rank of $\underset{m\times n}{\boldsymbol{A}}$ and is denoted $rank(\underset{m\times n}{\boldsymbol{A}})$ or $r(\underset{m\times n}{\boldsymbol{A}})$.

- If $r(\underset{m\times n}{A}) = m$, $\underset{m\times n}{A}$ is said to have full row rank.
- If $r(\underset{m\times n}{A}) = n$, $\underset{m\times n}{A}$ is said to have full column rank.

Inverse of a Matrix

- If $r(\underset{n\times n}{\boldsymbol{A}}) = n$, there exists a matrix $\underset{n\times n}{\boldsymbol{B}}$ such that $\underset{n\times n}{\boldsymbol{A}}\,\underset{n\times n}{\boldsymbol{B}} = \underset{n\times n}{\boldsymbol{I}}$.
- Such a matrix $\boldsymbol{B}$ is called the inverse of $\boldsymbol{A}$ and is denoted $\boldsymbol{A}^{-1}$.

- Prove that $r(\underset{n\times n}{\boldsymbol{A}}) = n \Longleftrightarrow \exists \underset{n\times n}{\boldsymbol{B}} \ni \underset{n\times n}{\boldsymbol{A}}\,\underset{n\times n}{\boldsymbol{B}} = \underset{n\times n}{\boldsymbol{I}}$.
- Prove that $\underset{n\times n}{\boldsymbol{A}}\,\underset{n\times n}{\boldsymbol{B}} = \underset{n\times n}{\boldsymbol{I}} \Longrightarrow \underset{n\times n}{\boldsymbol{B}}\,\underset{n\times n}{\boldsymbol{A}} = \underset{n\times n}{\boldsymbol{I}}$
- Thus $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$.

Singular / Nonsingular Matrix

- If $r(\underset{n\times n}{A}) = n$, $\underset{n\times n}{A}$ is said to be <u>nonsingular</u>.
- If $r(\underset{n\times n}{A}) < n$, $\underset{n\times n}{A}$ is said to be <u>singular</u>.

Inverse of a Nonsingular 2×2 Matrix

- $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ if $ad - bc \neq 0$.

- $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is singular if $ad - bc = 0$.

Column Space of a Matrix

The <u>column space</u> of a matrix $\underset{m\times n}{\boldsymbol{A}}$, denoted by $\mathcal{C}(\boldsymbol{A})$, is the vector space generated by the columns of $\underset{m\times n}{\boldsymbol{A}}$; i.e.,

$$\mathcal{C}(\boldsymbol{A}) = \{\boldsymbol{A}\boldsymbol{x} : \boldsymbol{x} \in \mathbb{R}^n\}.$$

$dim(\mathcal{C}(\boldsymbol{A})) = r(\underset{m \times n}{\boldsymbol{A}})$ because the largest possible subset of LI columns of $\underset{m \times n}{\boldsymbol{A}}$ is a basis for $\mathcal{C}(\boldsymbol{A})$.

- Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 4 \\ 4 & 2 & 8 \end{bmatrix}.$$

- Find $r(\boldsymbol{A})$.

- Give a basis for $\mathcal{C}(\boldsymbol{A})$.

- Characterize $\mathcal{C}(\boldsymbol{A})$.

Result A.1:

$$rank(\boldsymbol{AB}) \leq \min\{rank(\boldsymbol{A}), rank(\boldsymbol{B})\}$$

Proof of Result A.1:

- Let $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$ denote the columns of $\boldsymbol{B}$ so that $\boldsymbol{B} = [\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n]$.
- Then $\boldsymbol{AB} = [\boldsymbol{Ab}_1, \ldots, \boldsymbol{Ab}_n]$. This implies that the columns of $\boldsymbol{AB}$ are in $\mathcal{C}(\boldsymbol{A})$.

- $dim(\mathcal{C}(\boldsymbol{A}))$ is $rank(\boldsymbol{A})$.
- There does not exist a set of LI vectors in $\mathcal{C}(\boldsymbol{A})$ with more than $rank(\boldsymbol{A})$ vectors by Fact V2.
- It follows that $rank(\boldsymbol{AB}) \leq rank(\boldsymbol{A})$.

- It remains to show that

$$rank(\boldsymbol{AB}) \leq rank(\boldsymbol{B}).$$

- Note that $rank(\boldsymbol{AB})$ is the same as $rank((\boldsymbol{AB})') = rank(\boldsymbol{B}'\boldsymbol{A}')$.

- Our previous argument shows that

$$rank(\boldsymbol{B}'\boldsymbol{A}') \leq rank(\boldsymbol{B}') = rank(\boldsymbol{B}).$$

□

Provide an example where

$$rank(\boldsymbol{AB}) < \min\{rank(\boldsymbol{A}), rank(\boldsymbol{B})\}.$$

Result A.2:

(a) If $\boldsymbol{A} = \boldsymbol{BC}$, then $\mathcal{C}(\boldsymbol{A}) \subseteq \mathcal{C}(\boldsymbol{B})$,

(b) If $\mathcal{C}(\boldsymbol{A}) \subseteq \mathcal{C}(\boldsymbol{B})$, then there exist $\boldsymbol{C}$ such that $\boldsymbol{A} = \boldsymbol{BC}$.

Null Space of a Matrix

- The null space of a matrix $\boldsymbol{A}$, denoted $\mathcal{N}(\boldsymbol{A})$ is defined as

$$\mathcal{N}(\boldsymbol{A}) = \{\boldsymbol{y} : \boldsymbol{A}\boldsymbol{y} = \boldsymbol{0}\}.$$

- Note that $\mathcal{N}(\boldsymbol{A})$ is the set of vectors orthogonal to every row of $\boldsymbol{A}$.

A vector in $\mathcal{N}(\boldsymbol{A})$ can also be seen as a vector of coefficients corresponding to a LC of the columns of $\boldsymbol{A}$ that is $\mathbf{0}$.

Note that if $\boldsymbol{A}$ has dimension $m \times n$, then the vectors in $\mathcal{C}(\boldsymbol{A})$ have dimension m and the vectors in $\mathcal{N}(\boldsymbol{A})$ have dimension n.

Is the null space of a matrix $\underset{m \times n}{\boldsymbol{A}}$ a vector space?

Find the null space of

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \\ 3 & 6 \end{bmatrix}.$$

Result A.3:

$$rank(\underset{m\times n}{\boldsymbol{A}}) = n \iff \mathcal{N}(\boldsymbol{A}) = \{\boldsymbol{0}\}.$$

Theorem A.1:

If the matrix $\boldsymbol{A}$ is $m \times n$ with rank r, then

$$dim(\mathcal{N}(\boldsymbol{A})) = n - r,$$

or more elegantly,

$$dim(\mathcal{N}(\underset{m \times n}{\boldsymbol{A}})) + dim(\mathcal{C}(\underset{m \times n}{\boldsymbol{A}})) = n.$$

Proof of Theorem A.1:

- Let $k = dim(\mathcal{N}(\boldsymbol{A}))$. Results A.3 covers the case where $k = 0$. Suppose now that $k > 0$.

- Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$ form a basis for $\mathcal{N}(\boldsymbol{A})$. Then

$$\underset{m \times n}{\boldsymbol{A}} \boldsymbol{u}_i = \boldsymbol{0} \quad \forall\, i = 1, \ldots, k.$$

- By Fact V5, there exist $\boldsymbol{u}_{k+1}, \ldots, \boldsymbol{u}_n$ such that $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ form a basis for $\mathbb{R}^n$.

- We will now argue that the $n-k$ vectors $\boldsymbol{Au}_{k+1}, \ldots, \boldsymbol{Au}_n$ form a basis for $\mathcal{C}(\boldsymbol{A})$.

- If so, then

$$dim(\mathcal{N}(\boldsymbol{A})) + dim(\mathcal{C}(\boldsymbol{A})) = k + n - k = n,$$

i.e.,

$$dim(\mathcal{N}(\boldsymbol{A})) = n - dim(\mathcal{C}(\boldsymbol{A})) = n - r.$$

- First note that $\boldsymbol{A}\boldsymbol{u}_i \in \mathcal{C}(\boldsymbol{A}) \quad \forall\, i = k+1, \ldots, n.$
- Now note that

$$
\begin{aligned}
& c_{k+1}\boldsymbol{A}\boldsymbol{u}_{k+1} + \cdots + c_n\boldsymbol{A}\boldsymbol{u}_n = \boldsymbol{0} \\
\Longrightarrow\ & \boldsymbol{A}(c_{k+1}\boldsymbol{u}_{k+1} + \cdots + c_n\boldsymbol{u}_n) = \boldsymbol{0} \\
\Longrightarrow\ & c_{k+1}\boldsymbol{u}_{k+1} + \cdots + c_n\boldsymbol{u}_n \in \mathcal{N}(\boldsymbol{A}) \\
\Longrightarrow\ & \exists\ c_1, \ldots, c_k \in \mathbb{R} \ni c_1\boldsymbol{u}_1 + \cdots + c_k\boldsymbol{u}_k = \sum_{j=k+1}^{n} c_j\boldsymbol{u}_j \\
\Longrightarrow\ & c_1\boldsymbol{u}_1 + \cdots + c_k\boldsymbol{u}_k - c_{k+1}\boldsymbol{u}_{k+1} - \cdots - c_n\boldsymbol{u}_n = \boldsymbol{0} \\
\Longrightarrow\ & c_1 = \cdots = c_n = 0 \text{ by LI of } \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n.
\end{aligned}
$$

- Therefore, $\boldsymbol{Au}_{k+1}, \ldots, \boldsymbol{Au}_n$ are LI.
- Now let $\boldsymbol{U} = [\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n]$.
- Because $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ are LI and a basis for $\mathbb{R}^n$, $\exists\ \boldsymbol{U}^{-1} \ni \boldsymbol{U}\boldsymbol{U}^{-1} = \boldsymbol{I}$.
- Let $\boldsymbol{x} \in \mathbb{R}^n$ be arbitrary and define $\boldsymbol{z} = \boldsymbol{U}^{-1}\boldsymbol{x}$.

- Then

$$\begin{aligned} \boldsymbol{Ax} &= \boldsymbol{AUU}^{-1}\boldsymbol{x} = \boldsymbol{AUz} \\ &= [\boldsymbol{Au}_1, \ldots, \boldsymbol{Au}_k, \boldsymbol{Au}_{k+1}, \ldots, \boldsymbol{Au}_n]\boldsymbol{z} \\ &= [\boldsymbol{0}, \ldots, \boldsymbol{0}, \boldsymbol{Au}_{k+1}, \ldots, \boldsymbol{Au}_n]\boldsymbol{z} \\ &= z_{k+1}\boldsymbol{Au}_{k+1} + \cdots + z_n\boldsymbol{Au}_n. \end{aligned}$$

- Therefore, any vector in $\mathcal{C}(\boldsymbol{A})$ can be written as a LC of $\boldsymbol{Au}_{k+1}, \ldots, \boldsymbol{Au}_n$.

- It follows that

$$\boldsymbol{A}\boldsymbol{u}_{k+1}, \ldots, \boldsymbol{A}\boldsymbol{u}_n \text{ is a basis for } \mathcal{C}(\boldsymbol{A}).$$

- $\therefore n - k = r$ and $k + r = n$. $\square$