Stat 511 HW7 Solution

1. (a) $E(MS_{ou(xu,trt)})$

$$= E\left[\frac{1}{tnm-tn}\left(\sum_{i=1}^{t}\sum_{j=1}^{n}\sum_{k=1}^{m}(y_{ijk}-\bar{y}_{ij\cdot})^2\right)\right]$$

$$= \frac{1}{tnm-tn}\sum_{i=1}^{t}\sum_{j=1}^{n}\sum_{k=1}^{m}E(y_{ijk}-\bar{y}_{ij\cdot})^2$$

$$= \frac{1}{tn(m-1)}\sum_{i=1}^{t}\sum_{j=1}^{n}\sum_{k=1}^{m}E(\mu+\tau_i+u_{ij}+e_{ijk}-\mu-\tau_i-u_{ij}-\bar{e}_{ij\cdot})^2$$

$$= \frac{1}{tn(m-1)}\sum_{i=1}^{t}\sum_{j=1}^{n}\sum_{k=1}^{m}E(e_{ijk}-\bar{e}_{ij\cdot})^2$$

$$= \frac{1}{tn}\sum_{i=1}^{t}\sum_{j=1}^{n}E\left\{\frac{1}{m-1}\sum_{k=1}^{m}(e_{ijk}-\bar{e}_{ij\cdot})^2\right\}$$

$$= \frac{1}{tn}\sum_{i=1}^{t}\sum_{j=1}^{n}\sigma_e^2$$

$$= \frac{1}{tn}\cdot tn\,\sigma_e^2$$

$$= \sigma_e^2$$

1.(b). By slides 9 and 10,

$E(MS_{ou(xu,trt)})$

$$= E\left(\frac{\underset{\sim}{y}'(I-P_3)\underset{\sim}{y}}{tnm-tn}\right) = E\left[\underset{\sim}{y}'\frac{(I-P_3)}{tn(m-1)}\underset{\sim}{y}\right] \equiv E\left[\underset{\sim}{y}'A\underset{\sim}{y}\right],$$

where $A = \frac{I-P_3}{tn(m-1)}$

Then, $E(MS_{ou(xu,trt)})$

$$= tr(A\Sigma) + E(\underset{\sim}{y})'AE(\underset{\sim}{y})$$

$$A\Sigma = \frac{1}{tn(m-1)}(I-P_3)\left[\sigma_u^2 I_{tn\times tn} \otimes \underset{\sim}{1}\underset{\sim}{1}'_{m\times m} + \sigma_e^2 I_{tnm\times tnm}\right]$$

$$= \frac{1}{tn(m-1)}\left\{\sigma_u^2 I_{tn\times tn} \otimes \underset{\sim}{1}\underset{\sim}{1}'_{m\times m} + \sigma_e^2 I_{tnm\times tnm} - P_3\left[\sigma_u^2 I_{tn\times tn} \otimes \underset{\sim}{1}\underset{\sim}{1}'_{m\times m} + \sigma_e^2 I_{tnm\times tnm}\right]\right\} \quad \text{------ (1)}$$

Because P_3 is the projection matrix of $Z = I_{tn\times tn} \otimes \underset{\sim}{1}_{m\times 1}$,

and any column of $I_{tn\times tn} \otimes \underset{\sim}{1}\underset{\sim}{1}'_{m\times m}$ is a column of Z, i.e.,

$C(I_{tn\times tn} \otimes \underset{\sim}{1}\underset{\sim}{1}'_{m\times m}) \subset C(Z)$, we have,

$$P_3 \sigma_u^2 I_{tn\times tn} \otimes \underset{\sim}{1}\underset{\sim}{1}'_{m\times m}$$

$$= \sigma_u^2 P_3 I_{tn\times tn} \otimes \underset{\sim}{1}\underset{\sim}{1}'_{m\times m}$$

$$= \sigma_u^2 I_{tn\times tn} \otimes \underset{\sim}{1}\underset{\sim}{1}'_{m\times m} \quad \text{------ (2)}$$

Compute P_3 next.

$$P_3 = Z(Z'Z)^{-}Z'$$
$$= Z\left[(I_{tn\times tn} \otimes \underset{\sim}{1}'_{m\times 1})(I_{tn\times tn} \otimes \underset{\sim}{1}_{m\times 1})\right]^{-} Z'$$
$$= Z\left[I_{tn\times tn} \otimes (\underset{\sim}{1}'_{m\times 1}\, \underset{\sim}{1}_{m\times 1})\right]^{-} Z'$$
$$= Z\left[m\, I_{tn\times tn}\right]^{-1} Z'$$
$$= \frac{1}{m} Z Z'$$
$$= \frac{1}{m}(I_{tn\times tn} \otimes \underset{\sim}{1}_{m\times 1})(I_{tn\times tn} \otimes \underset{\sim}{1}'_{m\times 1})$$
$$= \frac{1}{m} I_{tn\times tn} \otimes \underset{\sim}{1}\underset{\sim}{1}'_{m\times m}$$

$$\therefore\ P_3 \sigma_e^2 I_{tnm\times tnm} = \sigma_e^2 P_3 = \frac{\sigma_e^2}{m} I_{tn\times tn} \otimes \underset{\sim}{1}\underset{\sim}{1}'_{m\times m} \quad \text{--- (3)}$$

By (1), (2), (3),

$$A\Sigma = \frac{1}{tn(m-1)}\left\{\sigma_e^2 I_{tnm\times tnm} - \frac{\sigma_e^2}{m} I_{tn\times tn} \otimes \underset{\sim}{1}\underset{\sim}{1}'_{m\times m}\right\}$$

$$\therefore\ tr(A\Sigma) = \frac{\sigma_e^2}{tn(m-1)}\left[tnm - \frac{1}{m}\cdot tnm\right]$$
$$= \sigma_e^2 \frac{1}{tn(m-1)} \cdot tn(m-1)$$
$$= \sigma_e^2$$

Now, consider $E(\underset{\sim}{y})' A\, E(\underset{\sim}{y})$.

$$E(\underset{\sim}{y})' A\, E(\underset{\sim}{y})$$
$$= \beta' X' \frac{(I-P_3)}{tn(m-1)} X\beta = \frac{1}{tn(m-1)} \beta' X' (I-P_3) X\beta = \frac{1}{tn(m-1)} \beta' X' (X - P_3 X)\beta$$
$$= 0 \quad \text{because } P_3 X = X \text{ since } C(X) \subset C(P_3)$$

Thus, $E(MS_{ou(xu,\,trt)})$

$$= tr(A\Sigma) + E(\underset{\sim}{Y})' A\, E(\underset{\sim}{Y})$$

$$= \sigma_e^2 + 0$$

$$= \sigma_e^2$$

1.(c). $\dfrac{MStrt}{1.5\sigma_u^2+\sigma_e^2} = \dfrac{(\bar{Y}_{1..}-\bar{Y}_{2..})^2}{1.5\sigma_u^2+\sigma_e^2}$, $\dfrac{MSxu(trt)}{\sigma_u^2+\sigma_e^2} = \dfrac{\frac{1}{2}(Y_{111}-Y_{121})^2}{\sigma_u^2+\sigma_e^2}$

Let $w_1 = \dfrac{\bar{Y}_{1..}-\bar{Y}_{2..}}{\sqrt{1.5\sigma_u^2+\sigma_e^2}}$, $W_2 = \dfrac{Y_{111}-Y_{121}}{\sqrt{2(\sigma_u^2+\sigma_e^2)}}$.

Then, $w_1^2 = \dfrac{MStrt}{1.5\sigma_u^2+\sigma_e^2}$, $W_2^2 = \dfrac{MSxu(trt)}{\sigma_u^2+\sigma_e^2}$.

W_1 and W_2 are normally distributed because each of them is a l.c. of normal R.V.'s.

Consider $E(W_1) = \dfrac{1}{\sqrt{1.5\sigma_u^2+\sigma_e^2}} E(\bar{Y}_{1..}-\bar{Y}_{2..})$

$$= \frac{1}{\sqrt{1.5\sigma_u^2+\sigma_e^2}}(E\bar{Y}_{1..} - E\bar{Y}_{2..})$$

$$= \frac{1}{\sqrt{1.5\sigma_u^2+\sigma_e^2}}(\mu+\tau_1-\mu-\tau_2)$$

$$= \frac{\tau_1-\tau_2}{\sqrt{1.5\sigma_u^2+\sigma_e^2}}$$

$$Var(W_1) = \frac{1}{1.5\sigma_u^2+\sigma_e^2} Var(\bar{Y}_{1..}-\bar{Y}_{2..})$$

$$= \frac{1}{1.5\sigma_u^2+\sigma_e^2} Var\left[\frac{1}{2}(Y_{111}+Y_{121}) - \frac{1}{2}(Y_{211}+Y_{212})\right]$$

$$= \frac{1}{1.5\sigma_u^2+\sigma_e^2}\cdot\frac{1}{4}\cdot\left[Var(Y_{111})+Var(Y_{121})+Var(Y_{211})+Var(Y_{212})+2cov(Y_{211},Y_{212})\right]$$

$$= \frac{1}{1.5\sigma_u^2+\sigma_e^2}\cdot\frac{1}{4}\cdot\left[4(\sigma_u^2+\sigma_e^2)+2\sigma_u^2\right]$$

$$= 1$$

$$\therefore\ W_1 \sim N\left(\frac{\tau_1-\tau_2}{\sqrt{1.5\sigma_u^2+\sigma_e^2}}, 1\right)$$

$$\therefore \frac{MS_{trt}}{1.5\sigma_u^2+\sigma_e^2} \sim \chi_1^2\left(\frac{(\tau_1-\tau_2)^2}{1.5\sigma_u^2+\sigma_e^2}\right) \qquad \text{--- (1)}$$

Similarly, $E(W_2) = \frac{1}{\sqrt{2(\sigma_u^2+\sigma_e^2)}} \left[E(Y_{111}) - E(Y_{121})\right]$

$$= \frac{1}{\sqrt{2(\sigma_u^2+\sigma_e^2)}}(\mu+\tau_1-\mu-\tau_1)$$

$$= 0$$

$$Var(W_2) = \frac{1}{2(\sigma_u^2+\sigma_e^2)} Var(Y_{111}-Y_{121})$$

$$= \frac{1}{2(\sigma_u^2+\sigma_e^2)}\left[Var(Y_{111}) + Var(Y_{121}) - 2cov(Y_{111}, Y_{121})\right]$$

$$= \frac{1}{2(\sigma_u^2+\sigma_e^2)}\left[2(\sigma_u^2+\sigma_e^2) - 0\right]$$

$$= 1$$

$$\therefore W_2 \sim N(0,1)$$

$$\therefore \frac{MS_{xu(trt)}}{\sigma_u^2+\sigma_e^2} \sim \chi_1^2(0) \qquad \text{--- (2)}$$

$$cov(W_1, W_2) = cov\left(\frac{\bar{Y}_{1..}-\bar{Y}_{2..}}{\sqrt{1.5\sigma_u^2+\sigma_e^2}}, \frac{Y_{111}-Y_{121}}{\sqrt{2(\sigma_u^2+\sigma_e^2)}}\right)$$

$$= \frac{1}{\sqrt{(1.5\sigma_u^2+\sigma_e^2)}\sqrt{2(\sigma_u^2+\sigma_e^2)}} cov\left(\frac{1}{2}(Y_{111}+Y_{121}) - \frac{1}{2}(Y_{211}+Y_{212}), Y_{111}-Y_{121}\right)$$

$$= \frac{1}{\sqrt{1.5\sigma_u^2+\sigma_e^2}\sqrt{2(\sigma_u^2+\sigma_e^2)}}\left\{Var(Y_{111}) - Var(Y_{121})\right\}/2$$

$$= \frac{1}{\sqrt{1.5\sigma_u^2+\sigma_e^2}\sqrt{2(\sigma_u^2+\sigma_e^2)}}(\sigma_u^2+\sigma_e^2-\sigma_u^2-\sigma_e^2)/2$$

$$= 0$$

$\therefore$ W_1, W_2 are independent

$\therefore$ $\dfrac{MS_{trt}}{1.5\sigma_u^2+\sigma_e^2}$, $\dfrac{MS_{xu(trt)}}{\sigma_u^2+\sigma_e^2}$ are independent ...... (3)

By (1), (2), (3),

$$\frac{MS_{trt}}{1.5\sigma_u^2+\sigma_e^2} \Big/ \frac{MS_{xu(trt)}}{\sigma_u^2+\sigma_e^2} \sim F_{1,1}\left(\frac{(\tau_1-\tau_2)^2}{1.5\sigma_u^2+\sigma_e^2}\right)$$

2.(a)

Source	D.f.	Sum of Square
Diet	$D-1$	$\sum_{i=1}^{D}\sum_{j=1}^{M}\sum_{k=1}^{L}(\bar{y}_{i\cdot\cdot}-\bar{y}_{\cdot\cdot\cdot})^2$
Drug	$M-1$	$\sum_{i=1}^{D}\sum_{j=1}^{M}\sum_{k=1}^{L}(\bar{y}_{\cdot j\cdot}-\bar{y}_{\cdot\cdot\cdot})^2$
Diet × Drug	$(D-1)(M-1)$	$\sum_{i=1}^{D}\sum_{j=1}^{M}\sum_{k=1}^{L}(\bar{y}_{ij\cdot}-\bar{y}_{i\cdot\cdot}-\bar{y}_{\cdot j\cdot}+\bar{y}_{\cdot\cdot\cdot})^2$
litter(Diet)	$D(L-1)$	$\sum_{i=1}^{D}\sum_{j=1}^{M}\sum_{k=1}^{L}(\bar{y}_{i\cdot k}-\bar{y}_{i\cdot\cdot})^2$
Error	$DML-DM-DL+D$	$\sum_{i=1}^{D}\sum_{j=1}^{M}\sum_{k=1}^{L}(y_{ijk}-\bar{y}_{ij\cdot}-\bar{y}_{i\cdot k}+\bar{y}_{i\cdot\cdot})^2$
Corrected Total	$DML-1$	$\sum_{i=1}^{D}\sum_{j=1}^{M}\sum_{k=1}^{L}(y_{ijk}-\bar{y}_{\cdot\cdot\cdot})^2$

The above notations are based on the model:

$$y_{ijk}=\mu+\alpha_i+\beta_j+\gamma_{ij}+u_{ik}+e_{ijk},$$

where $i=1,2 \cdots D$; $j=1,2,\cdots M$; $k=1,2,\cdots L$;

u_{ik} is the random effect of litters, u_{ik} iid $N(0,\sigma_u^2)$;

e_{ijk} iid $N(0,\sigma_e^2)$.

Note that "Error" corresponds to Drug × Litter(Diet), which has $(M-1)D(L-1)=DML-DM-DL+D$ degrees of freedom.

2.(b) $E(MS_{Diet})$

$$= \frac{1}{D-1} E\left\{ \sum_{i=1}^{D} \sum_{j=1}^{M} \sum_{k=1}^{L} (\bar{y}_{i..} - \bar{y}_{...})^2 \right\}$$

$$= \frac{1}{D-1} \cdot ML \cdot \sum_{i=1}^{D} E(\bar{y}_{i..} - \bar{y}_{...})^2$$

$$= \frac{ML}{D-1} \sum_{i=1}^{D} E\left[\mu + \alpha_i + \bar{\beta}_{.} + \bar{\gamma}_{i.} + \bar{u}_{i.} + \bar{e}_{i..} - \mu - \bar{\alpha}_{.} - \bar{\beta}_{.} - \bar{\gamma}_{..} - \bar{u}_{..} - \bar{e}_{...}\right]^2$$

$$= \frac{ML}{D-1} \left\{ \sum_{i=1}^{D} (\alpha_i - \bar{\alpha}_{.} + \bar{\gamma}_{i.} - \bar{\gamma}_{..})^2 + \sum_{i=1}^{D} E(\bar{u}_{i.} - \bar{u}_{..})^2 + \sum_{i=1}^{D} E(\bar{e}_{i..} - \bar{e}_{...})^2 \right\}$$

$$= \frac{ML}{D-1} \left[\sum_{i=1}^{D} (\alpha_i - \bar{\alpha}_{.} + \bar{\gamma}_{i.} - \bar{\gamma}_{..})^2 \right] + ML\, E\left[\frac{\sum_{i=1}^{D} (\bar{u}_{i.} - \bar{u}_{..})^2}{D-1} \right] + ML \cdot E\left[\frac{\sum_{i=1}^{D} (\bar{e}_{i..} - \bar{e}_{...})^2}{D-1} \right]$$

$$= \frac{ML}{D-1} \left[\sum_{i=1}^{D} (\alpha_i - \bar{\alpha}_{.} + \bar{\gamma}_{i.} - \bar{\gamma}_{..})^2 \right] + ML\, Var(\bar{u}_{i.}) + ML \cdot var(\bar{e}_{i..})$$

$$= \frac{ML}{D-1} \left[\sum_{i=1}^{D} (\alpha_i - \bar{\alpha}_{.} + \bar{\gamma}_{i.} - \bar{\gamma}_{..})^2 \right] + ML \cdot \frac{\sigma_u^2}{L} + ML \cdot \frac{\sigma_e^2}{ML}$$

$$= \frac{ML}{D-1} \left[\sum_{i=1}^{D} (\alpha_i - \bar{\alpha}_{.} + \bar{\gamma}_{i.} - \bar{\gamma}_{..})^2 \right] + M\sigma_u^2 + \sigma_e^2$$

$E(MS_{Drug})$

$$= \frac{1}{M-1} E\left\{ \sum_{i=1}^{D} \sum_{j=1}^{M} \sum_{k=1}^{L} (\bar{y}_{.j.} - \bar{y}_{...})^2 \right\}$$

$$= \frac{DL}{M-1} \sum_{j=1}^{M} E(\bar{y}_{.j.} - \bar{y}_{...})^2$$

$$= \frac{DL}{M-1} \sum_{j=1}^{M} E(\cancel{\mu} + \bar{\alpha}_{.} + \beta_j + \bar{\gamma}_{.j} + \bar{u}_{..} + \bar{e}_{.j.} \cancel{- \mu - \bar{\alpha}_{.}} - \bar{\beta}_{.} - \bar{\gamma}_{..} - \bar{u}_{..} - \bar{e}_{...})^2$$

$$= \frac{DL}{M-1} \sum_{j=1}^{M} E(\beta_j - \bar{\beta}_{.} + \bar{\gamma}_{.j} - \bar{\gamma}_{..} + \bar{e}_{.j.} - \bar{e}_{...})^2$$

$$= \frac{DL}{M-1}\sum_{j=1}^{M}(\beta_j - \bar{\beta}_. + \bar{\gamma}_{.j} - \bar{\gamma}_{..})^2 + \frac{DL}{M-1} E\left\{\sum_{j=1}^{M}(\bar{e}_{.j.} - \bar{e}_{...})^2\right\}$$

$$= \frac{DL}{M-1}\sum_{j=1}^{M}(\beta_j - \bar{\beta}_. + \bar{\gamma}_{.j} - \bar{\gamma}_{..})^2 + DL\, Var(\bar{e}_{.j.})$$

$$= \frac{DL}{M-1}\sum_{j=1}^{M}(\beta_j - \bar{\beta}_. + \bar{\gamma}_{.j} - \bar{\gamma}_{..})^2 + DL \cdot \frac{\sigma_e^2}{DL}$$

$$= \frac{DL}{M-1}\sum_{j=1}^{M}(\beta_j - \bar{\beta}_. + \bar{\gamma}_{.j} - \bar{\gamma}_{..})^2 + \sigma_e^2$$

$E(MS_{\text{Diet}\times\text{Drug}})$

$$= \frac{1}{(D-1)(M-1)} E\left\{\sum_{i=1}^{D}\sum_{j=1}^{M}\sum_{k=1}^{L}(\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...})^2\right\}$$

$$= \frac{L}{(D-1)(M-1)}\sum_{i=1}^{D}\sum_{j=1}^{M} E(\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...})^2$$

$$= \frac{L}{(D-1)(M-1)}\sum_{i=1}^{D}\sum_{j=1}^{M} E\big(\cancel{\alpha_i} + \cancel{\beta_j} + \gamma_{ij} \cancel{+ \bar{u}_{i.}} + \bar{e}_{ij.} \cancel{- \alpha_i} \cancel{- \bar{\beta}_.} - \bar{\gamma}_{i.} \cancel{- \bar{u}_{i.}} - \bar{e}_{i..}$$
$$\cancel{- \bar{\alpha}_.} \cancel{- \beta_j} - \bar{\gamma}_{.j} \cancel{- \bar{u}_{..}} - \bar{e}_{.j.} \cancel{+ \bar{\alpha}_.} \cancel{+ \bar{\beta}_.} + \bar{\gamma}_{..} \cancel{+ \bar{u}_{..}} + \bar{e}_{...}\big)^2$$

$$= \frac{L}{(D-1)(M-1)}\sum_{i=1}^{D}\sum_{j=1}^{M} E(\gamma_{ij} - \bar{\gamma}_{i.} - \bar{\gamma}_{.j} + \bar{\gamma}_{..} + \bar{e}_{ij.} - \bar{e}_{i..} - \bar{e}_{.j.} + \bar{e}_{...})^2$$

$$= \frac{L}{(D-1)(M-1)}\sum_{i=1}^{D}\sum_{j=1}^{M}(\gamma_{ij} - \bar{\gamma}_{i.} - \bar{\gamma}_{.j} + \bar{\gamma}_{..})^2 + L\cdot E\left\{\frac{1}{(D-1)(M-1)}\sum_{i=1}^{D}\sum_{j=1}^{M}(\bar{e}_{ij.} - \bar{e}_{i..} - \bar{e}_{.j.} + \bar{e}_{...})^2\right\}$$

$$= \frac{L}{(D-1)(M-1)}\sum_{i=1}^{D}\sum_{j=1}^{M}(\gamma_{ij} - \bar{\gamma}_{i.} - \bar{\gamma}_{.j} + \bar{\gamma}_{..})^2 + \frac{L}{(D-1)(M-1)}\sum_{i=1}^{D}\sum_{j=1}^{M} Var(\bar{e}_{ij.} - \bar{e}_{i..} - \bar{e}_{.j.} + \bar{e}_{...})$$

$---\ \bigstar$

where $var(\overline{e_{ij\cdot}} - \overline{e_{i\cdot\cdot}} - \overline{e_{\cdot j\cdot}} + \overline{e_{\cdots}})$

$= var(\overline{e_{ij\cdot}}) + Var(\bar{e}_{i\cdot\cdot}) + Var(\bar{e}_{\cdot j\cdot}) + Var(\bar{e}_{\cdots}) - 2cov(\overline{e_{ij\cdot}}, \overline{e_{i\cdot\cdot}})$

$-2cov(\overline{e_{ij\cdot}}, \overline{e_{\cdot j\cdot}}) + 2cov(\overline{e_{ij\cdot}}, \overline{e_{\cdots}}) - 2cov(\overline{e_{i\cdot\cdot}}, \overline{e_{\cdots}})$

$-2cov(\bar{e}_{\cdot j\cdot}, \overline{e_{\cdots}}) + 2cov(\overline{e_{i\cdot\cdot}}, \overline{e_{\cdot j\cdot}})$

$$= \frac{\sigma_e^2}{L} + \frac{\sigma_e^2}{ML} + \frac{\sigma_e^2}{DL} + \frac{\sigma_e^2}{DML} - 2\frac{1}{M}\frac{\sigma_e^2}{L} - \frac{2\sigma_e^2}{D\,L} + \frac{2\sigma_e^2}{DML}$$

$$- \frac{2\sigma_e^2}{DML} - \frac{2\sigma_e^2}{DML} + \frac{2\sigma_e^2}{DML}$$

$$= \sigma_e^2\left(\frac{1}{L} - \frac{1}{ML} - \frac{1}{DL} + \frac{1}{DML}\right)$$

$$= \sigma_e^2 (D-1)(M-1)/DML$$

By ✪, $E(MS_{Diet \times Drug})$

$$= \frac{L}{(D-1)(M-1)} \sum_{i=1}^{D}\sum_{j=1}^{M} (\gamma_{ij} - \bar{\gamma}_{i\cdot} - \bar{\gamma}_{\cdot j} + \bar{\gamma}_{\cdot\cdot})^2 + \frac{LDM}{(D-1)(M-1)} \frac{\sigma_e^2(D-1)(M-1)}{DML}$$

$$= \frac{L}{(D-1)(M-1)} \sum_{i=1}^{D}\sum_{j=1}^{M} (\gamma_{ij} - \bar{\gamma}_{i\cdot} - \bar{\gamma}_{\cdot j} + \bar{\gamma}_{\cdot\cdot})^2 + \sigma_e^2$$

$E(MS_{Litter(Diet)})$

$$= \frac{1}{D(L-1)} E\left\{\sum_{i=1}^{D}\sum_{j=1}^{M}\sum_{k=1}^{L} (\bar{y}_{i\cdot k} - \bar{y}_{i\cdot\cdot})^2\right\}$$

$$= \frac{1}{D(L-1)} M \sum_{i=1}^{D}\sum_{k=1}^{L} E(\bar{y}_{i\cdot k} - \bar{y}_{i\cdot\cdot})^2$$

$$= \frac{M}{D(L-1)} \sum_{i=1}^{D}\sum_{k=1}^{L} E(u_{ik} - \bar{u}_{i\cdot} + \bar{e}_{i\cdot k} - e_{i\cdot\cdot})^2$$

$$= \frac{M}{D} \sum_{i=1}^{D} \left[E\left\{ \frac{\sum_{k=1}^{L} (u_{ik} - \bar{u}_{i\cdot})^2}{L-1} \right\} + E\left\{ \frac{\sum_{k=1}^{L} (\bar{e}_{i\cdot k} - \bar{e}_{i\cdot\cdot})^2}{L-1} \right\} \right]$$

$$= \frac{M}{D} \sum_{i=1}^{D} \left\{ \sigma_u^2 + \frac{\sigma_e^2}{M} \right\}$$

$$= M\sigma_u^2 + \sigma_e^2$$

$E(MS_{Error})$

$$= \frac{1}{DML - DM - DL + D} E\left\{ \sum_{i=1}^{D} \sum_{j=1}^{M} \sum_{k=1}^{L} (y_{ijk} - \bar{y}_{ij\cdot} - \bar{y}_{i\cdot k} + \bar{y}_{i\cdot\cdot})^2 \right\}$$

$$= \frac{1}{D(M-1)(L-1)} \sum_{i=1}^{D} \sum_{j=1}^{M} \sum_{k=1}^{L} E(e_{ijk} - \bar{e}_{ij\cdot} - \bar{e}_{i\cdot k} + \bar{e}_{i\cdot\cdot})^2$$

$$= \frac{1}{D(M-1)(L-1)} \sum_{i=1}^{D} E\left\{ \sum_{j=1}^{M} \sum_{k=1}^{L} (e_{ijk} - \bar{e}_{ij\cdot} - \bar{e}_{i\cdot k} + \bar{e}_{i\cdot\cdot})^2 \right\}$$

$$= \frac{1}{D(M-1)(L-1)} \sum_{i=1}^{D} \left[(M-1)(L-1)\sigma_e^2 \right]$$

$$= \sigma_e^2$$

2.(c) $$\frac{SS_{Litter(Diet)}}{M\sigma_u^2+\sigma_e^2} \sim \chi^2_{D(L-1)}$$

$$\frac{SS_{Error}}{\sigma_e^2} \sim \chi^2_{D(M-1)(L-1)}$$

2.(d) To test for diet main effects, we can use

$$F = \frac{MS_{Diet}}{MS_{Litter(Diet)}},$$

which has a central F dist'n with $(D-1,\ D(L-1))$ d.f. under H_0.

2(e) To test for drug main effects, we can use

$$F = \frac{MS_{Drug}}{MS_{Error}},$$

which has a central F dist'n with $(M-1, D(M-1)(L-1))$ d.f. under H_0.

2.(f) To test for diet by drug interactions, we can use

$$F = \frac{MS_{Diet \times Drug}}{MS_{Error}},$$

which has a central F dist'n with $((D-1)(M-1), D(M-1)(L-1))$ d.f. under H_0.

2(g) The difference between the drug 1 mean and the drug 2 mean when averaging over diets is:

$$\beta_1 + \bar{\gamma}_{\cdot 1} - \beta_2 - \bar{\gamma}_{\cdot 2}$$

Consider $\bar{y}_{\cdot 1 \cdot} - \bar{y}_{\cdot 2 \cdot}$.

$$\bar{y}_{\cdot 1 \cdot} - \bar{y}_{\cdot 2 \cdot} = (\mu + \bar{\alpha}_{\cdot} + \beta_1 + \bar{\gamma}_{\cdot 1} + \bar{u}_{\cdot\cdot} + \bar{e}_{\cdot 1 \cdot}) - (\mu + \bar{\alpha}_{\cdot} + \beta_2 + \bar{\gamma}_{\cdot 2} + \bar{u}_{\cdot\cdot} + \bar{e}_{\cdot 2 \cdot})$$

$$= (\beta_1 + \bar{\gamma}_{\cdot 1} - \beta_2 - \bar{\gamma}_{\cdot 2}) + (\bar{e}_{\cdot 1 \cdot} - \bar{e}_{\cdot 2 \cdot})$$

$\therefore$ $\bar{y}_{\cdot 1 \cdot} - \bar{y}_{\cdot 2 \cdot} \sim N\left(\beta_1 + \bar{\gamma}_{\cdot 1} - \beta_2 - \bar{\gamma}_{\cdot 2}, \ \dfrac{2\sigma_e^2}{DL}\right)$

$\because$ $E(MS_{Error}) = \sigma_e^2$ and $\dfrac{MS_{Error} \cdot D(M-1)(L-1)}{\sigma_e^2} \sim \chi^2_{D(M-1)(L-1)}$ by (c)

$\therefore$ $\dfrac{(\bar{y}_{\cdot 1 \cdot} - \bar{y}_{\cdot 2 \cdot}) - (\beta_1 - \bar{\gamma}_{\cdot 1} - \beta_2 - \bar{\gamma}_{\cdot 2})}{\sqrt{\dfrac{2MS_{Error}}{DL}}} \sim t_{D(M-1)(L-1)}$

$\therefore$ A 95% confidence interval for the desired difference is

$$\left(\bar{y}_{\cdot 1 \cdot} - \bar{y}_{\cdot 2 \cdot} - t_{D(M-1)(L-1), 0.975}\sqrt{\frac{2MS_{Error}}{DL}}, \ \bar{y}_{\cdot 1 \cdot} - \bar{y}_{\cdot 2 \cdot} + t_{D(M-1)(L-1), 0.975}\sqrt{\frac{2MS_{Error}}{DL}}\right)$$

2(h) The difference between the diet 1 drug 1 mean and the diet 2 drug 1 mean is:

$\alpha_1 + \gamma_{11} - \alpha_2 - \gamma_{21}$

Consider $\bar{y}_{11\cdot} - \bar{y}_{21\cdot}$.

$$\bar{y}_{11\cdot} - \bar{y}_{21\cdot}$$
$$= (\mu + \alpha_1 + \beta_1 + \gamma_{11} + \bar{u}_{1\cdot} + \bar{e}_{11\cdot}) - (\mu + \alpha_2 + \beta_1 + \gamma_{21} + \bar{u}_{2\cdot} + \bar{e}_{21\cdot})$$
$$= (\alpha_1 + \gamma_{11} - \alpha_2 - \gamma_{21}) + (\bar{u}_{1\cdot} + \bar{e}_{11\cdot} - \bar{u}_{2\cdot} - \bar{e}_{21\cdot})$$

$$\text{var}(\bar{y}_{11\cdot} - \bar{y}_{21\cdot}) = \text{Var}(\bar{u}_{1\cdot} - \bar{u}_{2\cdot} + \bar{e}_{11\cdot} - \bar{e}_{21\cdot})$$
$$= \text{Var}(\bar{u}_{1\cdot} - \bar{u}_{2\cdot}) + \text{var}(\bar{e}_{11\cdot} - \bar{e}_{21\cdot})$$
$$= \frac{2\sigma_u^2}{L} + \frac{2\sigma_e^2}{L}$$

$$\therefore \bar{y}_{11\cdot} - \bar{y}_{21\cdot} \sim N\left(\alpha_1 + \gamma_{11} - \alpha_2 - \gamma_{21}, \frac{2(\sigma_u^2 + \sigma_e^2)}{L}\right)$$

Because $E(MS_{Litter(Diet)}) = M\sigma_u^2 + \sigma_e^2$ and $E(MS_{Error}) = \sigma_e^2$,

we have $\hat{\sigma}_e^2 = MS_{Error}$ and $\hat{\sigma}_u^2 = \frac{1}{M}(MS_{Litter(Diet)} - MS_{Error})$. (*)

Then, by Cochran-Satterthwaite approximation,

$$d \cdot \left[\frac{2(\hat{\sigma}_u^2 + \hat{\sigma}_e^2)}{L} \Big/ \frac{2(\sigma_u^2 + \sigma_e^2)}{L} \right] \sim \chi^2_d$$

where
$$d = \frac{\left\{ \frac{2}{L}\left[\frac{1}{M}(MS_{Litter(Diet)} - MS_{Error}) + MS_{Error}\right] \right\}^2}{\frac{\left[\frac{2}{L}\cdot\frac{1}{M}(MS_{Litter(Diet)})\right]^2}{D(L-1)} + \frac{\left[\frac{2}{L}(1-\frac{1}{M})MS_{Error}\right]^2}{D(M-1)(L-1)}}$$

Thus, $$\frac{\bar{y}_{11\cdot} - \bar{y}_{24\cdot} - (\alpha_1 + \gamma_{11} - \alpha_2 - \gamma_{21})}{\sqrt{\frac{2(\hat{\sigma}_u^2 + \hat{\sigma}_e^2)}{L}}} \sim t_d$$

$\therefore$ a 95% confidence interval for the desired difference is

$$\left(\bar{y}_{11\cdot} - \bar{y}_{24\cdot} - t_{d,0.975}\sqrt{\frac{2(\hat{\sigma}_u^2 + \hat{\sigma}_e^2)}{L}} \;,\; \bar{y}_{11\cdot} - \bar{y}_{24\cdot} + t_{d,0.975}\sqrt{\frac{2(\hat{\sigma}_u^2 + \hat{\sigma}_e^2)}{L}} \right)$$

where $\hat{\sigma}_u^2$ and $\hat{\sigma}_e^2$ are as in $(*)$

2(i) An average response for each litter is

$$\bar{y}_{i\cdot k} = \mu + \alpha_i + \bar{\beta}_\cdot + \bar{\gamma}_{i\cdot} + u_{ik} + \bar{e}_{i\cdot k},$$

for $i=1,\cdots,D$; $k=1,\cdots,L$.

2(j) Let $z_{ik} = \bar{y}_{i\cdot k}$, $\tau_i = \mu + \alpha_i + \bar{\beta}_{\cdot} + \bar{\gamma}_{i\cdot}$, $\varepsilon_{ik} = u_{ik} + \bar{e}_{i\cdot k}$,

then, $\varepsilon_{ik} \sim N(0, \sigma_u^2 + \frac{\sigma_e^2}{M})$.

By part (i),

$$z_{ik} = \tau_i + \varepsilon_{ik}, \quad \text{for } i = 1, \cdots, D;\ k = 1, \cdots, L.$$

This model has the form of the usual model for a completely randomized design.

The analysis using this model can give us the inference about the main effects of diets.

The results of such an analysis would be the same with those obtained using the full data set.