1. (a) Testing $H_0: C\beta = d$ under $y = X\beta + \varepsilon$, where $\varepsilon \sim N(0, \sigma^2 V)$ is equivalent with testing it under the model

$$z = W\beta + \delta, \text{ where } z = V^{-\frac{1}{2}} y,\ W = V^{-\frac{1}{2}} X,\ \delta = V^{-\frac{1}{2}} \varepsilon \quad (\star)$$

and thus $\delta \sim N(0, \sigma^2 I)$. The equivalency is because V is positive definite, thus, $\operatorname{rank}(W) = \operatorname{rank}(X)$.

Model $(\star)$ is a normal linear Gauss–Markov model, thus, the F-statistic for testing $H_0: C\beta = d$ is

$$F \equiv \frac{(C\hat{\beta} - d)' [C(W'W)^{-} C']^{-1} (C\hat{\beta} - d)/q}{\hat{\sigma}^2},$$

where $\hat{\beta} = (W'W)^{-} W' z = (X'V^{-1}X)^{-} X'V^{-1} y$; $q = \operatorname{rank}(C)$

$$\hat{\sigma}^2 = \frac{z'(I - P_W) z}{n - r}, \quad r = \operatorname{rank}(W) = \operatorname{rank}(X)$$

$$= \frac{y' V^{-\frac{1}{2}} (I - V^{-\frac{1}{2}} X (X'V^{-1}X)^{-} X' V^{-\frac{1}{2}}) V^{-\frac{1}{2}} y}{n - r} = \frac{y'(V^{-1} - V^{-1}X(X'V^{-1}X)^{-}X'V^{-1}) y}{n - r}$$

F has a noncentral F-distribution $F_{q, n-r}(\phi)$, where ϕ is

$$(C\beta - d)' [C(W'W)^{-} C']^{-1} (C\beta - d)/\sigma^2$$

$$= (C\beta - d)' [C(X'V^{-1}X)^{-} C']^{-1} (C\beta - d)/\sigma^2$$

1.(b): Model $\underset{\sim}{z} \sim N(W\underset{\sim}{\beta}, \sigma^2 I)$ is normal linear Gauss-Markov model, therefore,

$$\frac{\underset{\sim}{c}'\hat{\beta} - \underset{\sim}{c}'\underset{\sim}{\beta}}{\sqrt{\hat{\sigma}^2 \underset{\sim}{c}'(W'W)^- \underset{\sim}{c}}} \sim t_{n-r}$$

where $\hat{\sigma}^2 = \underset{\sim}{z}'(I-P_W)\underset{\sim}{z}/(n-r)$ $\qquad \hat{\beta} = (X'V^{-1}X)^- X'V^{-1}\underset{\sim}{y}$

Therefore, a $100(1-\alpha)\%$ confidence interval for $\underset{\sim}{c}'\underset{\sim}{\beta}$ is

$$\left(\underset{\sim}{c}'\hat{\beta} - t_{n-r,1-\frac{\alpha}{2}}\sqrt{\hat{\sigma}^2 \underset{\sim}{c}'(W'W)^- \underset{\sim}{c}}\ ,\ \underset{\sim}{c}'\hat{\beta} + t_{n-r,1-\frac{\alpha}{2}}\sqrt{\hat{\sigma}^2 \underset{\sim}{c}'(W'W)^- \underset{\sim}{c}}\right)$$

2. Let $\underset{\sim}{W} = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$,

Because $W_1 \sim N(4\mu, \sigma^2)$ and $W_2 \sim N(\mu, \sigma^2/3)$ are independent,

$\underset{\sim}{W} \sim N(\underset{\sim}{\mu}, \Sigma)$, where $\underset{\sim}{\mu} = \begin{bmatrix} 4\mu \\ \mu \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \mu$.

and $\Sigma = \begin{bmatrix} Var(W_1) & cov(W_1, W_2) \\ cov(W_1, W_2) & Var(W_2) \end{bmatrix} = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2/3 \end{bmatrix} = \sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}$

$\therefore \underset{\sim}{W} \sim N(X\mu, \sigma^2 V)$, where $X = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $V = \sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$

Thus, this is an Aitken model.

Therefore, the BLUE of μ is

$$\hat{\mu} = (X'V^{-1}X)^{-1} X' V^{-1} \underset{\sim}{W}$$

$$= \left(\begin{bmatrix} 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$$

$$= \frac{4}{19} W_1 + \frac{3}{19} W_2$$

3(a). $var(Y_i) = var(|X_i|\,\varepsilon_i)$

$$= |X_i|^2\, var(\varepsilon_i)$$

$$= X_i^2 \sigma^2$$

$$cov(|X_i| Y_i, |X_j| Y_j) = |X_i|\,|X_j|\, cov(Y_i, Y_j)$$

$$= |X_i|\,|X_j|\, 0$$

$$= 0$$

Thus, $var(Y) = \sigma^2\, diag(X_1^2, \cdots, X_n^2)$, i.e.,

$$Var(Y) = \sigma^2 \begin{bmatrix} X_1^2 & 0 & \cdots & \cdots & 0 \\ 0 & X_2^2 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & X_{n-1}^2 & 0 \\ 0 & \cdots & \cdots & 0 & X_n^2 \end{bmatrix}$$

3(b) $Var(\underset{\sim}{Y}) = \sigma^2 V$, where $V = diag(X_1^2, X_2^2, \cdots, X_n^2)$

Because $X_1, X_2, \cdots, X_n$ are known, V is known.

Thus, this is a special case of the Aitken model.

The design matrix is

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \equiv \underset{\sim}{x}$$ and the parameter vector is $\underset{\sim}{\beta} = [\beta]$ $(p=1)$

The BLUE is

$$(X'V^{-1}X)^{-1}X'V^{-1}\underset{\sim}{Y} = (\underset{\sim}{x}'V^{-1}\underset{\sim}{x})^{-1}\underset{\sim}{x}'V^{-1}\underset{\sim}{Y}$$

$$= \frac{\sum_{i=1}^{20} x_i Y_i / x_i^2}{\sum_{i=1}^{20} x_i^2 / x_i^2} = \frac{1}{n}\sum_{i=1}^{n}\frac{Y_i}{X_i}$$

4. (a) This is a split-plot experiment. The whole-plot experimental units are pots. The split-plot experimental units are seedlings.

(b) Seedlings.

(c)

Source	DF
watering level	$3-1=2$
pot (wat. lev.)	$(10-1)(3)=27$
injection	$2-1=1$
wat. lev. × injection	$(3-1)(2-1)=2$
error	87
c. total	$120-1=119$

Note that "error" is a combination of injection × pot (wat. lev.) $(2-1)(27)=27$ and seedling (injection, pot, wat. lev.) $=(2-1)(60)=60$

5.(a) $Y_{1j} - Y_{2j} = \mu_1 + u_j + e_{1j} - (\mu_2 + u_j + e_{2j})$

$$= \mu_1 - \mu_2 + e_{1j} - e_{2j}$$

$$E(Y_{1j} - Y_{2j}) = \mu_1 - \mu_2 + E(e_{1j}) - E(e_{2j})$$

$$= \mu_1 - \mu_2$$

$$var(Y_{1j} - Y_{2j}) = var(e_{1j} - e_{2j}) = Var(e_{1j}) + Var(e_{2j})$$

$$= 2\sigma_e^2$$

Thus, $d_1, \cdots, d_{20} \overset{iid}{\sim} N(\mu_1 - \mu_2, 2\sigma_e^2)$

(We have normality because linear combinations of normals are normal. We have independence because all e_{ij}'s are independent.)

5.(b) We should conduct a paired-data t-test.

Take $\underset{\sim}{y} = \underset{\sim}{d}$, $X = \underset{\sim}{1}$, $\underset{\sim}{\beta} = \mu_1 - \mu_2$, $\sigma^2 = 2\sigma_e^2$

$$\hat{\underset{\sim}{\beta}} = (X'X)^{-1}X'\underset{\sim}{y} = (\underset{\sim}{1}'\underset{\sim}{1})^{-1}\underset{\sim}{1}'\underset{\sim}{y} = \bar{y}_. = \bar{d}_.$$

$$Var(\hat{\underset{\sim}{\beta}}) = \sigma^2(X'X)^{-1} = 2\sigma_e^2/n = \sigma_e^2/10$$

$$\hat{\sigma}^2 = 2\hat{\sigma}_e^2 = \frac{(\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}})'(\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}})}{n - rank(X)}$$

$$= \frac{\sum_{i=1}^{20}(d_i - \bar{d}_.)^2}{19}$$

Thus, to test $H_0 : \mu_1 = \mu_2 \Leftrightarrow$ test $H_0 : \mu_1 - \mu_2 = 0$

we use $t = \dfrac{\bar{d}_.}{\sqrt{\hat{\sigma}^2/n}} = \dfrac{\bar{d}_.}{\sqrt{\dfrac{\sum_{i=1}^{20}(d_i - \bar{d}_.)^2}{19}/20}}$

5(c) Noncentral t with d.f. = 19 and noncentrality parameter

$$\frac{\mu_1-\mu_2}{\sqrt{\sigma^2/20}} = \frac{\mu_1-\mu_2}{\sqrt{2\sigma_e^2/20}} = \frac{\mu_1-\mu_2}{\sqrt{\sigma_e^2/10}}$$

This is true because

$$t = \frac{\dfrac{\bar{d}_\cdot-(\mu_1-\mu_2)}{\sqrt{\sigma^2/20}} + \dfrac{\mu_1-\mu_2}{\sqrt{\sigma^2/20}}}{\sqrt{\dfrac{\hat{\sigma}^2}{\sigma^2}}}$$

$$\frac{\bar{d}_\cdot-(\mu_1-\mu_2)}{\sqrt{\sigma^2/20}} \sim N(0,1)$$

independent of

$$\frac{\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{19}/19$$

5(d). This is the classic case of two independent normal samples. Take

$$\underset{\sim}{y} = \begin{bmatrix} \underset{\sim}{a} \\ \underset{\sim}{b} \end{bmatrix}, \quad X = \begin{bmatrix} \underset{20\times1}{\underset{\sim}{1}} & \underset{20\times1}{\underset{\sim}{0}} \\ \underset{20\times1}{\underset{\sim}{0}} & \underset{20\times1}{\underset{\sim}{1}} \end{bmatrix}, \quad \underset{\sim}{\beta} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$

$$\sigma^2 = var(u_j + e_{ij}) = \sigma_u^2 + \sigma_e^2$$

$$\hat{\underset{\sim}{\beta}} = (X'X)^{-1}X'\underset{\sim}{y} = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix}^{-1} \begin{bmatrix} a. \\ b. \end{bmatrix} = \begin{bmatrix} \bar{a}. \\ \bar{b}. \end{bmatrix}$$

$$\underset{\sim}{c}'\hat{\underset{\sim}{\beta}} = [1, -1] \begin{bmatrix} \bar{a}. \\ \bar{b}. \end{bmatrix} = \bar{a}. - \bar{b}.$$

$$Var(\underset{\sim}{c}'\hat{\underset{\sim}{\beta}}) = \underset{\sim}{c}'\sigma^2(X'X)^{-1}\underset{\sim}{c} = \sigma^2 [1, -1] \begin{bmatrix} \frac{1}{20} & 0 \\ 0 & \frac{1}{20} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{\sigma^2}{10}$$

$$\hat{\sigma}^2 = \frac{(\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}})'(\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}})}{n - rank(X)}$$

$$= \frac{\sum_{i=1}^{20}(a_i - \bar{a}.)^2 + \sum_{i=1}^{20}(b_i - \bar{b}.)^2}{38}$$

Thus, the formula for the interval is

$$\bar{a}. - \bar{b}. \pm t_{38}^{(.975)} \sqrt{\frac{\sum_{i=1}^{20}(a_i - \bar{a}.)^2 + \sum_{i=1}^{20}(b_i - \bar{b}.)^2}{(38)(10)}}$$

5(e). From previous parts we have

$$E\left(\frac{\sum_{i=1}^{20}(d_i-\bar{d}_.)^2}{19}\right)=2\sigma_e^2$$

and $$E\left(\frac{\sum_{i=1}^{20}(d_i-\bar{d}_.)^2+\sum_{i=1}^{20}(b_i-\bar{b}_.)^2}{38}\right)=\sigma_u^2+\sigma_e^2$$

Thus, $$\hat{\sigma}_e^2=\frac{1}{2}\,\frac{\sum_{i=1}^{20}(d_i-\bar{d}_.)^2}{19}$$

and $$\hat{\sigma}_u^2=\frac{\sum_{i=1}^{20}(a_i-\bar{a}_.)^2+\sum_{i=1}^{20}(b_i-\bar{b}_.)^2}{38}-\frac{1}{2}\,\frac{\sum_{i=1}^{20}(d_i-\bar{d}_.)^2}{19}$$

5(f). You could set the problem up as

$$\underset{\sim}{y} = \begin{bmatrix} \bar{d}. \\ \bar{a}. \\ \bar{b}. \end{bmatrix}, \quad X = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \underset{\sim}{\beta} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$

$$\underset{\sim}{\varepsilon} \sim N\left(\underset{\sim}{0}, \underbrace{\begin{bmatrix} \frac{\sigma_e^2}{10} & 0 & 0 \\ 0 & \frac{\sigma_u^2+\sigma_e^2}{20} & 0 \\ 0 & 0 & \frac{\sigma_u^2+\sigma_e^2}{20} \end{bmatrix}}_{\Sigma}\right)$$

You could then compute

$$(X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}\underset{\sim}{y}$$

Alternatively, we know that $\bar{d}.$ and $\bar{a}.-\bar{b}.$ are independent estimators of $\mu_1-\mu_2$. The BLUP will be of the form

$$\alpha\bar{d}. + (1-\alpha)(\bar{a}.-\bar{b}.)$$

with weights inversely proportional to the variances of $\bar{d}.$ and $\bar{a}.-\bar{b}.$

$$var(\bar{d}.) = \frac{2\sigma_e^2}{20} = \frac{\sigma_e^2}{10}$$

$$Var(\bar{a}.-\bar{b}.) = \frac{\sigma_u^2+\sigma_e^2}{10}$$

Thus,

$$\alpha = \frac{10/\sigma_e^2}{10/\sigma_e^2 + \dfrac{10}{\sigma_u^2+\sigma_e^2}} = \frac{10(\sigma_u^2+\sigma_e^2)}{10(\sigma_u^2+\sigma_e^2)+10\,\sigma_e^2} = \frac{\sigma_u^2+\sigma_e^2}{\sigma_u^2+2\sigma_e^2}$$

The BLUP is, therefore,

$$\frac{\sigma_u^2+\sigma_e^2}{\sigma_u^2+2\sigma_e^2}\,\bar{d}_. + \frac{\sigma_e^2}{\sigma_u^2+2\sigma_e^2}\,(\bar{a}_. - \bar{b}_.)$$