STAT 511 Homework 4

Due Date: 11:00 A.M., Wednesday, February 8

1. Suppose $\boldsymbol{w} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\boldsymbol{\mu} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \quad and \quad \boldsymbol{\Sigma} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix}$$

(a) Determine the distribution of

$$\begin{bmatrix} w_1 + w_2 - w_3 \\ w_1 + w_2 + w_3 \end{bmatrix}.$$

(b) Determine the distribution of

$$\frac{w_1^2 + w_2^2 + w_3^2 + 2w_1w_2 + 2w_1w_3 + 2w_2w_3}{16}.$$

(c) Determine the distribution of

$$\frac{4w_1^2 + 4w_2^2 + 4w_3^2 + 8w_1w_2 - 8w_1w_3 - 8w_2w_3}{w_1^2 + w_2^2 + w_3^2 + 2w_1w_2 + 2w_1w_3 + 2w_2w_3}.$$

2. If a random variable $w \sim N(\delta, 1)$ and is independent of a random variable $u \sim \chi^2_n$, then

$$\frac{w}{\sqrt{u/n}}$$

has a noncentral t-distribution with noncentrality parameter (ncp) δ and degrees of freedom (df) n. We will denote this distribution by $t_n(\delta)$. If the ncp is 0, the distribution is known as the central t-distribution, which can be denoted by $t_n(0)$ or t_n. Let $t_{n,1-\alpha}$ denote the $1-\alpha$ quantile of the $t_n(0)$ distribution. Thus, $t_{n,1-\alpha}$ is the value such that

$$P[T \leq t_{n,1-\alpha}] = 1 - \alpha$$

for $T \sim t_n(0)$.

For the following questions, suppose the Normal Theory Gauss-Markov linear model holds, and suppose $\boldsymbol{c}'\boldsymbol{\beta}$ is estimable.

(a) For a fixed constant d, prove that the distribution of

$$\frac{\boldsymbol{c}'\hat{\boldsymbol{\beta}} - d}{\sqrt{\hat{\sigma}^2 \boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^- \boldsymbol{c}}}$$

is noncentral t and derive the ncp and df.

(b) Provide a test statistic for testing the null hypothesis

$$H_0 : \boldsymbol{c}'\boldsymbol{\beta} = d \text{ vs. } H_A : \boldsymbol{c}'\boldsymbol{\beta} \neq d$$

for any fixed $d \in \mathbb{R}$.

(c) Suppose you wish to test the hypotheses in (b) using significance level α. For what values of the test statistic provided in (b) would you reject the null hypothesis?

(d) Derive expressions for the lower and upper limits of a $100(1-\alpha)$ confidence interval for $\boldsymbol{c}'\boldsymbol{\beta}$.

3. Prove the result stated on slide 3 of the notes on estimation of the error variance under the Gauss-Markov model.