Stat 511 HW3 Solution

1.(a) Because $\text{rank}(AB) \le \min\{\text{rank}(A), \text{rank}(B)\}$ and $P_X X = X$, we have

$$
\begin{aligned}
\text{rank}(X) &= \text{rank}(P_X X) \\
&= \text{rank}(X(X'X)^- X'X) \\
&\le \text{rank}(X'X) \\
&\le \text{rank}(X)
\end{aligned}
$$

$\therefore \ \text{rank}(X) = \text{rank}(X'X)$

(b) Similarly,

$$\begin{aligned} rank(X) &= rank(P_X X) \qquad\qquad P_X X = X \\ &\le rank(P_X) = rank(X(X'X)^{-}X') \\ &\le rank(X) \qquad\qquad \text{By } rank(AB) \le \min\{rank(A), rank(B)\} \end{aligned}$$

$\therefore \quad rank(X) = rank(P_X)$

(c) $q = \text{rank}(C)$ since C has q rows, which are L.I.

$= \text{rank}(AX)$ since $C\beta$ is estimable, $\exists$ a matrix $\underset{q\times n}{A} \ni C = AX$

$= \text{rank}(AP_X X)$ Since $P_X X = X$

$\leq \text{rank}(AP_X)$ since $\text{rank}(AP_X X) \leq \min\{\text{rank}(AP_X), \text{rank}(X)\}$

$= \text{rank}[(AP_X)']$ since $\text{rank}(X) = \text{rank}(X')$ $\forall X$

$= \text{rank}\{[(AP_X)']'(AP_X)'\}$ By part (a)

$= \text{rank}[AP_X (AP_X)']$

$= \text{rank}[AP_X P_X' A']$

$= \text{rank}[AP_X P_X A']$ Since $P_X' = P_X$

$= \text{rank}(AP_X A')$ Since $P_X P_X = P_X$

$= \text{rank}(AX(X'X)^- X'A')$

$= \text{rank}(C(X'X)^- C')$ since $C = AX$

$\leq \text{rank}(C)$ By the property that $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$

$= q$.

$\therefore$ $\text{rank}(C(X'X)^- C') = q$

2. (a) The design matrix X is

$$X = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

(b) i. Note that $\mu + s_1 = E(Y_{121})$.

By Result 3.7 (i) on Page 198, $\mu + s_1$ is estimable

ii. Note that $\beta = (\mu + s_1) - (\mu + s_1 - \beta) = E(Y_{121}) - E(Y_{111})$,

thus, by Result 3.7, β is estimable.

By Part (i), $\mu + s_1$ is also estimable, thus, $\mu + s_1 + 10\beta$ is a l.c. of estimable functions. By Result 3.7 (ii), $\mu + s_1 + 10\beta$ is estimable.

iii. Note that $s_1 - s_2 = (\mu + s_1) - (\mu + s_2) = E(Y_{121}) - E(Y_{221})$,

thus, by Result 3.7, $s_1 - s_2$ is estimable.

iv. To prove that μ is non-estimable, we will use the property that $\underset{\sim}{c}'\underset{\sim}{\beta}$ is estimable if and only if $\underset{\sim}{c}'\underset{\sim}{a} = 0$ $\forall$ $\underset{\sim}{a}$ for which $X\underset{\sim}{a} = 0$

Note that $X\underset{\sim}{a} = 0 \Rightarrow \underset{\sim}{a} = [1 \ -1 \ -1 \ 0]' \cdot b$, where $b \in \mathbb{R}$

Let $\underset{\sim}{a} = [1 \ -1 \ -1 \ 0]'$.

Then, $X\underset{\sim}{a} = 0$, but $\underset{\sim}{c}'\underset{\sim}{a} = (1 \ 0 \ 0 \ 0)\begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix} = 1 \neq 0$

Therefore, μ is non-estimable.

(c) A full-column-rank matrix that has the same column space as X in Part (a) is

$$W = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

(d) Using W as a design matrix, we get an equivalent model $\underset{\sim}{y} = W\underset{\sim}{\alpha} + \underset{\sim}{\varepsilon}$, where $\underset{\sim}{\alpha} = [\tau_1 \ \tau_2 \ \beta]'$, where $\tau_1 = \mu + s_1$ $\tau_2 = \mu + s_2$.

Then we have $E(Y_{111}) = \tau_1 - \beta = \underset{\sim}{c}'\underset{\sim}{\alpha}$, where $\underset{\sim}{c} = (1 \ 0 \ -1)'$

Note that the BLUE of $\underset{\sim}{c}'\underset{\sim}{\alpha}$ is $\underset{\sim}{c}'\hat{\underset{\sim}{\alpha}}_{OLS} = \underset{\sim}{c}'(W'W)^{-1}W'\underset{\sim}{y}$,

i.e., the BLUE of $E(Y_{111})$ is $\underset{\sim}{c}'(W'W)^{-1}W'\underset{\sim}{y}$.

We then computes the following

$$(W'W)^{-1} = \begin{bmatrix} \frac{1}{6} & & \\ & \frac{1}{6} & \\ & & \frac{1}{8} \end{bmatrix} \quad \text{and} \quad W'\underset{\sim}{y} = \begin{bmatrix} y_{1\cdot\cdot} \\ y_{2\cdot\cdot} \\ y_{\cdot 3\cdot} - y_{\cdot 1\cdot} \end{bmatrix},$$

where $y_{1\cdot\cdot} = \sum_{j=1}^{3}\sum_{k=1}^{2} y_{1jk}$, $y_{2\cdot\cdot} = \sum_{j=1}^{3}\sum_{k=1}^{2} y_{2jk}$, $y_{\cdot 3\cdot} = \sum_{i=1}^{2}\sum_{k=1}^{2} y_{i3k}$

and $y_{\cdot 1\cdot} = \sum_{i=1}^{2}\sum_{k=1}^{2} y_{i1k}$.

$$\therefore \hat{\underset{\sim}{\alpha}}_{OLS} = (W'W)^{-1}W'\underset{\sim}{y} = \begin{bmatrix} \bar{y}_{1\cdot\cdot} \\ \bar{y}_{2\cdot\cdot} \\ \frac{1}{2}(\bar{y}_{\cdot 3\cdot} - \bar{y}_{\cdot 1\cdot}) \end{bmatrix},$$

where $\bar{y}_{i\cdot\cdot} = \frac{1}{6} y_{i\cdot\cdot}$ for $i = 1, 2$.

$\bar{y}_{\cdot j\cdot} = \frac{1}{4} y_{\cdot j\cdot}$ for $j = 1, 2, 3$.

Therefore, the BLUE of $E(Y_{111})$ is $(1 \ 0 \ -1)'\hat{\underset{\sim}{\alpha}}_{OLS}$

$$= \bar{y}_{1\cdot\cdot} - \frac{1}{2}(\bar{y}_{\cdot 3\cdot} - \bar{y}_{\cdot 1\cdot})$$

(e) i. $\widehat{(\mu+s_1)} = \hat{\tau}_1 = \bar{y}_{1\cdot\cdot}$

ii. $\widehat{(\mu+s_1+10\beta)} = \hat{\tau}_1 + 10\hat{\beta} = \bar{y}_{1\cdot\cdot} + 5(\bar{y}_{\cdot 3\cdot} - \bar{y}_{\cdot 1\cdot})$

iii. $\widehat{(s_1-s_2)} = \hat{\tau}_1 - \hat{\tau}_2 = \bar{y}_{1\cdot\cdot} - \bar{y}_{2\cdot\cdot}$