Stat 511 HW 2 Solution

1. Pf: ($\Rightarrow$) part:

If $A=0$, then $A'A=0$

($\Leftarrow$) part:

If $A'A=0$, then $tr(A'A)=0$

By problem 6 on HW1, this implies $A=0$

2. Pf:

($\Leftarrow$): $XA = XB \Rightarrow X'XA = X'XB$ trivially

($\Rightarrow$): $X'XA = X'XB \Rightarrow X'XA - X'XB = 0$

$$\Rightarrow X'X(A-B) = 0$$

$$\Rightarrow (A-B)'X'X(A-B) = 0$$

$$\Rightarrow [X(A-B)]'X(A-B) = 0$$

$$\Rightarrow X(A-B) = 0 \qquad \text{by Problem 1}$$

$$\Rightarrow XA = XB$$

3. Pf:

By the definition of generalized inverse, we have

$$X'X(X'X)^{-}X'X = X'X = X'XI$$

By the result of problem 2, this implies

$$X(X'X)^{-}X'X = XI = X$$

i.e. $X(X'X)^{-}X'X = X$

4. Pf:

Since G is a generalized inverse of A,

we have $AGA = A$

Since A is symmetric, i.e., $A = A'$

thus, $A = A' = (AGA)'$

$= A'G'A'$

$= AG'A$ since $A = A'$

$\therefore$ G' is also a generalized inverse of A.

5. Pf:

First note that $(X'X)' = X'X$

$\therefore$ $X'X$ is symmetric

By the result of Problem 4, we have

$[(X'X)^-]'$ is also a generalized inverse of $X'X$.

Then, by the result of Problem 3, we have

$$X[(X'X)^-]'X'X = X$$

$$\Rightarrow X'X\{[(X'X)^-]'\}'X' = X'$$

$$\Rightarrow X'X(X'X)^- X' = X'$$

6. <u>Pf:</u> $P_X P_X = \underline{X(X'X)^- X'X}(X'X)^- X'$

$= X(X'X)^- X'$ By the result of Problem 3

$= P_X$

Altenatively,

$P_X P_X = X(X'X)^- \underline{X'X(X'X)^- X'}$

$= X(X'X)^- X'$ By the result of Problem 5

$= P_X$

$\therefore P_X$ is idempotent

7. Pf: $P_X' = [X(X'X)^- X']'$

$= X[(X'X)^-]'X'.$

Because $X'X$ is symmetric and $(X'X)^-$ is a generalized inverse of $X'X$,

by the result of Problem 4, $[(X'X)^-]'$ is also a generalized inverse of $X'X$

Since $P_X = X(X'X)^- X'$ is the same matrix no matter which generalized inverse of $X'X$ is used,

we have $P_X' = X[(X'X)^-]'X' = X(X'X)^- X' = P_X$

i.e., P_X is symmetric.

8. <u>Pf:</u> Note $(\underset{\sim}{y} - P_x \underset{\sim}{y})'(P_x \underset{\sim}{y} - \underset{\sim}{z})$

$= (\underset{\sim}{y}' - \underset{\sim}{y}' P_x)(P_x \underset{\sim}{y} - \underset{\sim}{z})$ $\qquad P_x' = P_x$

$= \underset{\sim}{y}' P_x \underset{\sim}{y} - \underset{\sim}{y}' P_x \underset{\sim}{y} - \underset{\sim}{y}' \underset{\sim}{z} + \underset{\sim}{y}' P_x \underset{\sim}{z}$ $\qquad P_x P_x = P_x$

$= -\underset{\sim}{y}' \underset{\sim}{z} + \underset{\sim}{y}' \underset{\sim}{z}$ $\qquad P_x \underset{\sim}{z} = \underset{\sim}{z}$ since $\underset{\sim}{z} \in C(X)$

$= 0$

Also note $P_x \underset{\sim}{y} - \underset{\sim}{z} \neq 0$ since $\underset{\sim}{z} \neq P_x \underset{\sim}{y}$

$\therefore$ By Hint:

$\|\underset{\sim}{y} - \underset{\sim}{z}\|^2 = \|\underset{\sim}{y} - P_x \underset{\sim}{y} + P_x \underset{\sim}{y} - \underset{\sim}{z}\|^2$

$> \|\underset{\sim}{y} - P_x \underset{\sim}{y}\|^2$

$\therefore \|\underset{\sim}{y} - \underset{\sim}{z}\| > \|\underset{\sim}{y} - P_x \underset{\sim}{y}\|$

9. Pf: By the result of Problem 8, we have

$Q(\underset{\sim}{b}) = \|\underset{\sim}{y} - X\underset{\sim}{b}\|^2 \geq \|\underset{\sim}{y} - P_X \underset{\sim}{y}\|^2$,

where the equality holds if and only if

$\underset{\sim}{b} = \underset{\sim}{b}^*$, where $\underset{\sim}{b}^*$ is the solution of the equation

$X\underset{\sim}{b} = P_X \underset{\sim}{y}$

Because P_X projects onto $C(X)$, we know $P_X \underset{\sim}{y} \in C(X)$

Thus, $\exists$ at least one vector $\underset{\sim}{b} \ni P_X \underset{\sim}{y} = X\underset{\sim}{b}$

We can take $\underset{\sim}{b}^*$ to be any such vector $\underset{\sim}{b}$, Then,

we have shown that $\underset{\sim}{b}^*$ will minimize $Q(\underset{\sim}{b})$ over $\underset{\sim}{b} \in \mathbb{R}^p$

Note $X\underset{\sim}{b} = P_X \underset{\sim}{y} \iff X\underset{\sim}{b} = X(X'X)^- X'\underset{\sim}{y}$

$\iff X'X\underset{\sim}{b} = X'X(X'X)^- X'\underset{\sim}{y}$ by Problem 2

$\iff X'X\underset{\sim}{b} = X'\underset{\sim}{y}$ by Problem 5

Thus, $Q(\underset{\sim}{b}^*) \leq Q(\underset{\sim}{b})$ if and only if $\underset{\sim}{b}^*$ is the solution to

the normal equation $X'X\underset{\sim}{b} = X'\underset{\sim}{y}$.

10. Since the last row of A is the sum of the first three rows and the first three rows L.I., we have $\text{rank}(A) = 3$

Note one of the 3×3 nonsingular submatrix of A is

$$W = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 9 & 4 \\ 0 & 1 & 4 \end{bmatrix}$$

The inverse of $\begin{bmatrix} 9 & 4 \\ 1 & 4 \end{bmatrix}$ is $\frac{1}{36-4}\begin{bmatrix} 4 & -4 \\ -1 & 9 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{32} & \frac{9}{32} \end{bmatrix}$

$$\therefore W^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{8} & -\frac{1}{8} \\ 0 & -\frac{1}{32} & \frac{9}{32} \end{bmatrix}$$

$$\therefore (W^{-1})' = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{8} & -\frac{1}{32} \\ 0 & -\frac{1}{8} & \frac{9}{32} \end{bmatrix}$$

Now replace the matrix W in A with $(W^{-1})'$ and all the other elements in A with zeros, we get

$$\begin{bmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{8} & -\frac{1}{32} & 0 \\ 0 & -\frac{1}{8} & \frac{9}{32} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then, transpose the above matrix, we get a G.I. of A:

$$G = \begin{bmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{8} & -\frac{1}{8} & 0 \\ 0 & -\frac{1}{32} & \frac{9}{32} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

11. (a) The design matrix is

$$X = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

(b) $\mu + \beta_2 = [1, 0, 0, 0, 1] \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \underset{\sim}{c}'\underset{\sim}{\beta}$

Note $X \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = 0$, but $[1 \ 0 \ 0 \ 0 \ 1] \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = 1 \neq 0$

$\therefore \exists \underset{\sim}{a} \ni X\underset{\sim}{a} = 0$ but $\underset{\sim}{c}'\underset{\sim}{a} \neq 0$

$\therefore \underset{\sim}{c}'\underset{\sim}{\beta}$ is nonestimable by Result 3.8 (ii) on Page 201

(c) Note $\beta_1 - \beta_2 = (\mu + \alpha_1 + \beta_1) - (\mu + \alpha_1 + \beta_2) = E(y_{111}) - E(y_{121})$, which is a l.c. of expectations of two observations.

Thus, by Result 3.7 on Page 198, $\beta_1 - \beta_2$ is estimable

(d) To get the same model, we only need to find a design matrix W that has the same column space with X. For instance, either

$W = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ or $W = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ would work.