STAT 511 Homework 2

Due Date: 11:00 A.M., Wednesday, January 25

The main purpose of the first eight questions on this homework is to prove the results about orthogonal project matrices stated on slide 11 of the notes entitled *Estimation of the Response Mean.* If you complete problems 1 through 8, you will have demonstrated all those properties from basic principles without using a circular argument.

1. Prove that a matrix $\boldsymbol{A}$ is $\mathbf{0}$ if and only if $\boldsymbol{A}'\boldsymbol{A} = \mathbf{0}$. (The proof is very short and follows easily from problem 6 on the first homework assignment.)

2. Prove that $\boldsymbol{X}'\boldsymbol{X}\boldsymbol{A} = \boldsymbol{X}'\boldsymbol{X}\boldsymbol{B}$ if and only if $\boldsymbol{X}\boldsymbol{A} = \boldsymbol{X}\boldsymbol{B}$. (Note that the "if" part of the proof, i.e.,

$$\boldsymbol{X}\boldsymbol{A} = \boldsymbol{X}\boldsymbol{B} \implies \boldsymbol{X}'\boldsymbol{X}\boldsymbol{A} = \boldsymbol{X}'\boldsymbol{X}\boldsymbol{B},$$

holds trivially. Thus, proving the converse, i.e.,

$$\boldsymbol{X}'\boldsymbol{X}\boldsymbol{A} = \boldsymbol{X}'\boldsymbol{X}\boldsymbol{B} \implies \boldsymbol{X}\boldsymbol{A} = \boldsymbol{X}\boldsymbol{B},$$

is the challenging part. One proof starts like this

$$\begin{aligned} \boldsymbol{X}'\boldsymbol{X}\boldsymbol{A} = \boldsymbol{X}'\boldsymbol{X}\boldsymbol{B} \quad &\implies \quad \boldsymbol{X}'\boldsymbol{X}\boldsymbol{A} - \boldsymbol{X}'\boldsymbol{X}\boldsymbol{B} = \mathbf{0} \\ &\implies \quad \boldsymbol{X}'\boldsymbol{X}(\boldsymbol{A} - \boldsymbol{B}) = \mathbf{0} \end{aligned}$$

Now if you multiply on the left by the appropriate matrix, you can use the result of problem 1 to help complete the proof.)

3. Use the definition of generalized inverse and the result of problem 2 to prove that

$$\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{X} = \boldsymbol{X}$$

for any $(\boldsymbol{X}'\boldsymbol{X})^{-}$ a generalized inverse of $\boldsymbol{X}'\boldsymbol{X}$.

4. Prove that if $\boldsymbol{A}$ is any symmetric matrix and $\boldsymbol{G}$ is any generalized inverse of $\boldsymbol{A}$, then it must be true that $\boldsymbol{G}'$ is also a generalized inverse of $\boldsymbol{A}$.

5. Use the results of problems 3 and 4 to prove that

$$\boldsymbol{X}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}' = \boldsymbol{X}'$$

for any $(\boldsymbol{X}'\boldsymbol{X})^{-}$ a generalized inverse of $\boldsymbol{X}'\boldsymbol{X}$.

6. Show that idempotency of $\boldsymbol{P_X}$ (i.e., $\boldsymbol{P_X}\boldsymbol{P_X} = \boldsymbol{P_X}$) follows from the result of problem 3 and, alternatively, from the result of problem 5.

7. In class notes, we argued that $\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'$ is the same matrix no matter which generalized inverse of $\boldsymbol{X}'\boldsymbol{X}$ is used. That argument depended only on the results of problem 3 and 5. Explain why this together with the result of problem 4 implies that $\boldsymbol{P_X}$ is symmetric (i.e., $\boldsymbol{P}'_{\boldsymbol{X}} = \boldsymbol{P_X}$).

8. Suppose $\boldsymbol{X}$ is an $n \times p$ design matrix and $\boldsymbol{y}$ is an $n \times 1$ vector. Suppose $\boldsymbol{z} \in \boldsymbol{C}(\boldsymbol{X})$ and $\boldsymbol{z} \neq \boldsymbol{P_X y}$. Prove that $\|\boldsymbol{y} - \boldsymbol{z}\| > \|\boldsymbol{y} - \boldsymbol{P_X y}\|$. Hint: Note that for an vector $\boldsymbol{a}$ and any vector $\boldsymbol{b} \neq \boldsymbol{0}$ such that $\boldsymbol{a}'\boldsymbol{b} = 0$

$$\begin{aligned} \|\boldsymbol{a}+\boldsymbol{b}\|^2 &= (\boldsymbol{a}+\boldsymbol{b})'(\boldsymbol{a}+\boldsymbol{b}) = (\boldsymbol{a}'+\boldsymbol{b}')(\boldsymbol{a}+\boldsymbol{b}) \\ &= \boldsymbol{a}'\boldsymbol{a} + \boldsymbol{a}'\boldsymbol{b} + \boldsymbol{b}'\boldsymbol{a} + \boldsymbol{b}'\boldsymbol{b} = \boldsymbol{a}'\boldsymbol{a} + 2\boldsymbol{a}'\boldsymbol{b} + \boldsymbol{b}'\boldsymbol{b} \\ &= \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 + 2\boldsymbol{a}'\boldsymbol{b} \\ &= \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 \text{ (because } \boldsymbol{a}'\boldsymbol{b} = 0). \\ &> \|\boldsymbol{a}\|^2 \text{ (because } \boldsymbol{b} \neq \boldsymbol{0}). \end{aligned}$$

Now note that

$$\|\boldsymbol{y} - \boldsymbol{z}\|^2 = \|\boldsymbol{y} - \boldsymbol{P_X y} + \boldsymbol{P_X y} - \boldsymbol{z}\|^2 = \dots$$

9. Prove the result in the first bullet on slide 26 of the slides entitled *Estimation of the Response Mean*. Hint: Consider a strategy similar to that used in problem 8.

10. Find a generalized inverse of the following matrix.

$$\boldsymbol{A} = \begin{bmatrix} 3 & 0 & 0 & 3 \\ 0 & 9 & 4 & 13 \\ 0 & 1 & 4 & 5 \\ 3 & 10 & 8 & 21 \end{bmatrix}$$

11. Read Ken Koehler's slides 188-207 on estimability. Now consider the Gauss-Markov Linear Model for the special case of $E(y_{ijk}) = \mu + \alpha_i + \beta_j, i = 1, 2; j = 1, 2; k = 1, 2$.

   (a) Write down the design matrix $\boldsymbol{X}$.

   (b) Is $\mu + \beta_2$ estimable? Show why or why not.

   (c) Is $\beta_1 - \beta_2$ estimable? Show why or why not.

   (d) Write down a different design matrix than in part (a) that will result in the same model for $y_{ijk}, i = 1, 2; j = 1, 2; k = 1, 2$.