Stat 511 HW1 Solution

1. Let $A$ be the matrix. Then, using basic row and column transformations, we can find the rank of this matrix:

$$\text{rank}(A) = \text{rank}\left(\begin{bmatrix} 1 & \frac{2}{3} & \frac{7}{3} & \frac{1}{3} & \frac{5}{3} \\ -1 & 3 & 5 & 1 & 3 \\ 4 & 2 & 8 & 7 & -6 \end{bmatrix}\right) \quad \text{row}1 \times \frac{1}{3}$$

$$= \text{rank}\left(\begin{bmatrix} 1 & \frac{2}{3} & \frac{7}{3} & \frac{1}{3} & \frac{5}{3} \\ 0 & \frac{11}{3} & \frac{22}{3} & \frac{4}{3} & \frac{14}{3} \\ 0 & -\frac{2}{3} & -\frac{4}{3} & \frac{17}{3} & -\frac{38}{3} \end{bmatrix}\right) \quad \begin{matrix}\text{row}2 + \text{row}1 \\ \text{row}3 - 4\,\text{row}1\end{matrix}$$

$$= \text{rank}\left(\begin{bmatrix} 1 & \frac{2}{3} & \frac{7}{3} & \frac{1}{3} & \frac{5}{3} \\ 0 & 0 & 0 & \frac{65}{2} & -65 \\ 0 & -\frac{2}{3} & -\frac{4}{3} & \frac{17}{3} & -\frac{38}{3} \end{bmatrix}\right) \quad \text{row}2 + \frac{11}{2}\,\text{row}3$$

$$= \text{rank}\left(\begin{bmatrix} 1 & \frac{2}{3} & \frac{7}{3} & \frac{1}{3} & \frac{5}{3} \\ 0 & -\frac{2}{3} & -\frac{4}{3} & \frac{17}{3} & -\frac{38}{3} \\ 0 & 0 & 0 & \frac{65}{2} & -65 \end{bmatrix}\right) \quad \text{exchange row 2 ; row3}$$

$$= 3$$

2. (a) Let $X$ be the design matrix, then,

$\text{rank}(X) \leq \min(n\text{row}(X), n\text{col}(X)) = 2$,

where $n\text{row}(X)$ and $n\text{col}(X)$ are the number of rows and number of columns of $X$ respectively. Thus, the largest possible rank of this matrix is 2.

To achieve the maximum rank, columns of $X$ must be linearly independent, i.e., $x_1, x_2, \cdots x_8$ can not be all the same.

3. (a) Using the same method that was demonstrated in Problem 1, we can find the rank of the matrix. Take the following transformations on the matrix:

$$\text{col}\,1 = \text{col}\,1 - \text{col}\,2 - \text{col}\,3$$
$$\text{col}\,2 = \text{col}\,2 - \text{col}\,6 - \text{col}\,7$$
$$\text{col}\,3 = \text{col}\,3 - \text{col}\,8 - \text{col}\,9$$
$$\text{col}\,4 = \text{col}\,4 - \text{col}\,6 - \text{col}\,8$$
$$\text{col}\,5 = \text{col}\,5 - \text{col}\,7 - \text{col}\,9$$

Then, we get:

$$\begin{bmatrix} \underset{8\times1}{0} & \underset{8\times1}{0} & \underset{8\times1}{0} & \underset{8\times1}{0} & \underset{8\times1}{0} & \begin{matrix}\underset{2\times1}{1}\\ \underset{6\times1}{0}\end{matrix} & \begin{matrix}\underset{2\times1}{0}\\ \underset{2\times1}{1}\\ \underset{4\times1}{0}\end{matrix} & \begin{matrix}\underset{4\times1}{0}\\ \underset{2\times1}{1}\\ \underset{2\times1}{0}\end{matrix} & \begin{matrix}\underset{6\times1}{0}\\ \underset{2\times1}{1}\end{matrix} \end{bmatrix} \quad (*)$$

where $\underset{a\times b}{0}$ stands for a matrix of dimension $a\times b$ with elements all 0;

$\underset{a\times b}{1}$ stands for a matrix of dimension $a\times b$ with elements all 1.

Thus, the rank of the design matrix is 4.

Another way to do the problem is to note that the last four columns are LI, and that each of the first 5 columns can be written as a linear combination of the last 4.

(b) The matrix (*) above has the same column space as the design matrix.

4. (a) ① Suppose $\underset{\sim}{x} \in S$, $a \in \mathbb{R}$, then

$$\underset{\sim}{1}'\underset{\sim}{x} = 0$$

$$\therefore \underset{\sim}{1}'(a\underset{\sim}{x}) = a\underset{\sim}{1}'\underset{\sim}{x} = 0$$

$$\therefore a\underset{\sim}{x} \in S$$

$\therefore S$ is closed under scalar multiplication

② Suppose $\underset{\sim}{x_1}, \underset{\sim}{x_2} \in S$, then

$$\underset{\sim}{1}'\underset{\sim}{x_1} = 0 \qquad \underset{\sim}{1}'\underset{\sim}{x_2} = 0$$

$$\Rightarrow \underset{\sim}{1}'(\underset{\sim}{x_1} + \underset{\sim}{x_2}) = \underset{\sim}{1}'\underset{\sim}{x_1} + \underset{\sim}{1}'\underset{\sim}{x_2} = 0 + 0 = 0$$

$\therefore S$ is closed under addition

by ① and ②: $S$ is a vector space

(b) Suppose $\underset{\sim}{x} = (x_1, x_2, x_3)' \in S$

i.e., $\underset{\sim}{1}'\underset{\sim}{x} = x_1 + x_2 + x_3 = 0$

$$\therefore x_3 = -x_1 - x_2$$

$$\therefore \underset{\sim}{x} = (x_1, x_2, -x_1 - x_2)' = (x_1, 0, -x_1)' + (0, x_2, -x_2)'$$

$$= x_1(1, 0, -1)' + x_2(0, 1, -1)'$$

Note that $(1, 0, -1)' \in S$ $(0, 1, -1)' \in S$ since $\underset{\sim}{1}'(1, 0, -1)' = 0$

and $\underset{\sim}{1}'(0, 1, -1)' = 0$

Also, $(1, 0, -1)'$ and $(0, 1, -1)'$ are L.I.

$$\therefore \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right) = S$$

$\therefore \left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\}$ forms a basis for $S$

(c) By part (b), the dimension of $S$ is 2.

5. $\text{cov}(A\underset{\sim}{z}+\underset{\sim}{a}, B\underset{\sim}{z}+\underset{\sim}{b})$

$$= E[\{A\underset{\sim}{z}+\underset{\sim}{a}-E(A\underset{\sim}{z}+\underset{\sim}{a})\}\{B\underset{\sim}{z}+\underset{\sim}{b}-E(B\underset{\sim}{z}+\underset{\sim}{b})\}']$$

$$= E[\{A\underset{\sim}{z}+\underset{\sim}{a}-AE(\underset{\sim}{z})-\underset{\sim}{a}\}\{B\underset{\sim}{z}+\underset{\sim}{b}-BE(\underset{\sim}{z})-\underset{\sim}{b}\}']$$

$$= E[A\{\underset{\sim}{z}-E(\underset{\sim}{z})\}\{B(\underset{\sim}{z}-E(\underset{\sim}{z}))\}']$$

$$= E[A\{\underset{\sim}{z}-E(\underset{\sim}{z})\}\{\underset{\sim}{z}-E(\underset{\sim}{z})\}'B']$$

$$= AE[\{\underset{\sim}{z}-E(\underset{\sim}{z})\}\{\underset{\sim}{z}-E(\underset{\sim}{z})]']B'$$

$$= A\,\text{cov}(\underset{\sim}{z},\underset{\sim}{z})B'$$

$$= A\,\text{var}(\underset{\sim}{z})B'$$

$$= A\Sigma B'$$

6. Pf: "$\Rightarrow$":

since $A=0$, we have $A'A=0$

$\therefore tr(A'A)=0$

"$\Leftarrow$":

Let $\underset{\sim}{a}_k$, $k=1,2,\cdots,n$ be the kth column of $\underset{m\times n}{A}$, then

$$tr(A'A)=\sum_{k=1}^{n}\underset{\sim}{a}_k'\underset{\sim}{a}_k=0$$

This implies that $\underset{\sim}{a}_k=\underset{\sim}{0}$ $\forall k=1,2,\cdots n$

$\therefore A=0$

7. (a) By the Property (iv) on Page 49 of Koehler's notes,

$$|diag(a_1,a_2,a_3,a_4)|=|diag(a_1,a_2)|\,|diag(a_3,a_4)|$$
$$=|a_1|\cdot|a_2|\cdot|a_3|\cdot|a_4|$$
$$=a_1a_2a_3a_4$$

(b) Let $A$ stand for $diag(a_1,a_2,a_3,a_4)$

Then, $|A-\lambda I|=0$

$\Leftrightarrow |diag(a_1-\lambda,a_2-\lambda,a_3-\lambda,a_4-\lambda)|=0$

i.e. $(\lambda-a_1)(\lambda-a_2)(\lambda-a_3)(\lambda-a_4)=0$

$\therefore$ The eigenvalues of $A$ are $a_1$, $a_2$, $a_3$ and $a_4$

8. Pf: $tr(B^{-1}AB)=tr(ABB^{-1})$ By the cyclic property of the trace

$=tr(A)$

9. Pf: Suppose $\underset{\sim}{x}_i$ is the eigenvector of $A$ corresponding to $\lambda_i$, then

(iv) $\underset{\sim}{x}_i'A\underset{\sim}{x}_i>0$ Since $A$ P.D. (Positive Definite)

i.e. $\underset{\sim}{x}_i'\lambda_i\underset{\sim}{x}_i>0$ $\forall i=1,2,\cdots k$ (because $A\underset{\sim}{x}_i=\lambda\underset{\sim}{x}_i$)

$\therefore \lambda_i\underset{\sim}{x}_i'\underset{\sim}{x}_i>0$ $\forall i=1,2,\cdots k$

$\therefore \lambda_i > 0 \quad \forall i = 1, 2, \cdots k \qquad$ (because $\underset{\sim}{x}_i \neq 0 \Rightarrow \underset{\sim}{x}_i' \underset{\sim}{x}_i > 0$)

(vi) The result is a special case of property (ii) on Page 49 of KK's Notes. Another way of showing it is as follows:

let $X \equiv [\underset{\sim}{x}_1, \cdots, \underset{\sim}{x}_k]$

Then, $|A||X| = |AX| = |[A\underset{\sim}{x}_1, A\underset{\sim}{x}_2, \cdots A\underset{\sim}{x}_k]|$

$= |[\lambda_1 \underset{\sim}{x}_1, \lambda_2 \underset{\sim}{x}_2, \cdots \lambda_k \underset{\sim}{x}_k]|$

$= |X \, diag(\lambda_1, \cdots \lambda_k)|$

$= |X| \, |diag(\lambda_1, \cdots \lambda_k)|$

$= |X| \prod_{i=1}^{k} \lambda_i = \prod_{i=1}^{k} \lambda_i \, |X|$

Now, multiply on right by $|X'|$

$$|A| \, |X| \, |X|' = \prod_{i=1}^{k} \lambda_i \, |X| |X'| \qquad (\Delta)$$

Note that $|X| \, |X|' = |XX|' = |I| = 1 \qquad$ since $X'X = I \Rightarrow XX' = I$

$\therefore$ By $(\Delta)$: $\quad |A| = \prod_{i=1}^{k} \lambda_i$

(vii) Suppose $\lambda$ is an eigenvalue of $A$, then for its eigenvector $\underset{\sim}{x} \neq \underset{\sim}{0}$

$\lambda \underset{\sim}{x} = A\underset{\sim}{x} = AA\underset{\sim}{x} = A(\lambda \underset{\sim}{x}) = \lambda A \underset{\sim}{x} = \lambda^2 \underset{\sim}{x}$

$\therefore (\lambda^2 - \lambda) \underset{\sim}{x} = \underset{\sim}{0}$

$\therefore \lambda^2 - \lambda = 0 \qquad$ since $\underset{\sim}{x} \neq \underset{\sim}{0}$

$\therefore \lambda = 0$ or $1$

10. $tr(A) = tr(UDU') = tr(DU'U) \qquad$ by cyclic property

$= tr(DI) = tr(D)$

$= \sum_{i=1}^{k} \lambda_i$

11. If $\begin{bmatrix} 6 \\ 1 \\ -2 \end{bmatrix}$ in the span of $\left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ 0 \\ 8 \end{bmatrix} \right\}$, then $\exists\ x_1, x_2 \in \mathbb{R}$ such that

$$\begin{bmatrix} 6 \\ 1 \\ -2 \end{bmatrix} = x_1 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} a \\ 0 \\ 8 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 6 = 4x_1 + ax_2 \\ 1 = x_1 \\ -2 = 8x_2 \end{cases}$$

$$\Rightarrow a = -8$$

12. $$\mathrm{var}\left( \begin{bmatrix} W_1 + W_2 \\ W_1 - W_2 \end{bmatrix} \right) = \begin{pmatrix} \mathrm{Var}(W_1 + W_2) & \mathrm{cov}(W_1 + W_2, W_1 - W_2) \\ \mathrm{Cov}(W_1 + W_2, W_1 - W_2) & \mathrm{Var}(W_1 - W_2) \end{pmatrix} \quad (\triangle)$$

$$\mathrm{Var}(W_1 + W_2) = \mathrm{Var}(W_1) + \mathrm{Var}(W_2) + 2 \cdot \mathrm{cov}(W_1, W_2)$$
$$= 4 + 2 - 2 = 4$$

$$\mathrm{Var}(W_1 - W_2) = \mathrm{var}(W_1) + \mathrm{var}(W_2) - 2\mathrm{cov}(W_1, W_2)$$
$$= 4 + 2 + 2 = 8$$

$$\mathrm{Cov}(W_1 + W_2, W_1 - W_2) = \mathrm{cov}(W_1, W_1) + \mathrm{cov}(W_2, W_1) - \mathrm{cov}(W_1, W_2) - \mathrm{cov}(W_2, W_2)$$
$$= \mathrm{var}(W_1) - \mathrm{var}(W_2)$$
$$= 4 - 2$$
$$= 2$$

$$\therefore \text{by } (\triangle),\ \mathrm{var}\left( \begin{bmatrix} W_1 + W_2 \\ W_1 - W_2 \end{bmatrix} \right) = \begin{bmatrix} 4 & 2 \\ 2 & 8 \end{bmatrix}$$